

Appendix F.

We will construct an example of a regular space which is not completely regular commonly referred to as the *Tychonoff corkscrew*.

We begin by presenting some background material related to ordinal numbers.

Recall that, if α is a limit ordinal, $\alpha = [0, \alpha)$ and $\alpha \cup \{\alpha\} = \alpha + 1 = [0, \alpha]$. Let ω_1 denote the first uncountable ordinal. It is also described as being the set of all countable ordinals. Then $\omega_1 = [0, \omega_1)$. We represent the first infinite ordinal by ω_0 . It is also described as the set of all finite ordinals, hence

$$\omega_0 = [0, 1, 2, 3, \dots) = \mathbb{N}$$

Recall that

$$\begin{aligned}\omega_1 \cup \{\omega_1\} &= \omega_1 + 1 = [0, \omega_1] \\ \omega_0 \cup \{\omega_0\} &= \omega_0 + 1 = [0, \omega_0]\end{aligned}$$

Fact #1: Every continuous real-valued function on ω_1 is constant on a tail-end of ω_1 (see page 516).

Fact #2: No countable subset is cofinal in ω_1 (see page 107).

Let $S = [0, \omega_1] \times [0, \omega_0]$. This set is referred to as the *Tychonoff plank*.

The set

$$T = S \setminus \{(\omega_1, \omega_0)\}$$

when equipped with the product topology is referred to as the *deleted Tychonoff plank*.

Fact #3: The deleted Tychonoff plank, T , is C^* -embedded in the Tychonoff plank $S = [0, \omega_1] \times [0, \omega_0]$, so $\beta T = S$. (See Theorem 21.18)

The set

$$\{\omega_1\} \times \omega_0 = \{(\omega_1, n) : n \in \mathbb{N}\}$$

is referred to the *right edge of T*.

The set

$$\omega_1 \times \{\omega_0\} = \{(\alpha, \omega_0) : \alpha \in \omega_1\}$$

is referred to the *top edge of T*.

Some notation.

The set \mathbb{Z} denotes the collection of all integers. Consider the product space,

$$Z = T \times \mathbb{Z}$$

The set Z can be viewed as a collection of horizontal planes each homeomorphic to T .

Then,

$$Z = \{ \cdots, \{T \times \{-3\}, T \times \{-2\}, T \times \{-1\}, T \times \{0\}, T \times \{1\}, T \times \{2\}, \cdots, \}$$

For each $n \in \mathbb{Z}$, we define

$$T_n = T \times \{n\}$$

the plane, homeomorphic to T “at level n ”.

A partition of Z .

We partition the set Z as follows:

$$A_1 = \{ [\{\omega_1\} \times \omega_0] \times \{n\}, [\{\omega_1\} \times \omega_0] \times \{n+1\} \} : n \text{ is even, point for point } \}$$

(For example, we collapse the *right edge*, $\{\omega_1\} \times \omega_0$, of the plane T at level 4 to the right edge of T at level 5, point by point.)

$$A_2 = \{ [\omega_1 \times \{\omega_0\}] \times \{n\}, [\omega_1 \times \{\omega_0\}] \times \{n+1\} \} : n \text{ is odd, point for point } \}$$

(For example, we collapse the *top edge*, $\omega_1 \times \{\omega_0\}$, of the plane T at level 3 to the top edge of T at level 4, point by point.)

All other points of T remain fixed under q .

Let \mathcal{D} denote the resulting decomposition of the space Z .

Let $q : Z \rightarrow \mathcal{D}$ be the quotient map associating each point in Z to the unique element in \mathcal{D} which contains it.

Equip \mathcal{D} with the quotient topology τ_q to obtain the quotient space (\mathcal{D}, τ_q) induced by q . Then U is open in \mathcal{D} provided that $q^{-1}[U]$ is open Z .

We introduce the following notation:

$$T_{q(n)} = q[T \times \{n\}]$$

See that q maps $T \times \{n\}$ homeomorphically onto $T_{q(n)}$.

For example, on the top edge, at level 3,

$$T_{q(3)}[\omega_1 \times \{\omega_0\}] \times \{3\} = T_{q(4)}[\omega_1 \times \{\omega_0\}] \times \{4\}$$

For the right edge, at level 4,

$$T_{q(4)}[\{\{\omega_1\} \times \omega_0\} \times \{4\}] = T_{q(5)}[\{\{\omega_1\} \times \omega_0\} \times \{5\}]$$

Finally, we add two points a^+ and a^- to the decomposition space \mathcal{D} .

The basic neighborhoods of a^+ are sets of the form

$$B_n(a^+) = \{a^+\} \cup \{T_{q(n+1)}, T_{q(n+2)}, T_{q(n+3)}, \dots\}$$

The basic neighborhoods of a^- are sets of the form

$$B_n(a^-) = \{a^-\} \cup \{T_{q(-n-1)}, T_{q(-n-2)}, T_{q(-n-3)}, \dots\}$$

We represent the resulting space $\mathcal{D} \cup \{a^-, a^+\}$ equipped with the described topology as

$$\mathcal{K} = \cup\{T_{q(n)} : n \in \mathbb{Z}\} \cup \{a^-, a^+\}$$

Lemma F1. Let $\mathcal{K} = \cup\{T_{q(n)} : n \in \mathbb{Z}\} \cup \{a^-, a^+\}$ be the set with topology τ described above.

The space (\mathcal{K}, τ) is a regular space.

Proof: Recall that both spaces, $[0, \omega_0]$ and $[0, \omega_1]$, are normal spaces (see the example on page 233). Then, by Theorem 9.15, the Tychonoff plank, $[0, \omega_1] \times [\omega_0]$ is normal, hence is regular. Since subspaces of regular spaces are regular (by Theorem 9.12) then the *deleted Tychonoff plank* T is regular. In what follows if n is odd, then

$$T_{q(n)}^* = T_{q(n)} \setminus \text{right edge of } T_{q(n)}$$

Case 1. Let F be a closed subset of \mathcal{K} which does not contain the point a^+ . Suppose

$$F \subseteq U = \cup\{T_{q(6)}^*, T_{q(4)}, T_{q(3)}, \dots\}$$

and

$$a^+ \in V = \cup\{T_{q(7)}^*, T_{q(8)}, T_{q(9)}, \dots\} \cup \{a^+\}$$

Then V and U are disjoint open neighborhoods of F and a^+ , respectively. Proceed similarly for F and a^- .

Case 2. Suppose $x \in T_{q(m)}$ and F is a closed subset of \mathcal{K} .

It is easily verified that there exists an open neighborhood V of x (in \mathcal{K}) such that $\text{cl}_{\mathcal{K}} V \cap F = \emptyset$.

Then \mathcal{K} is regular.

Lemma F2. Let $f : \mathcal{K} \rightarrow \mathbb{R}$ be a continuous real-valued function on \mathcal{K} .

The right edge of $T \times \{e\}$ is the set $\{(\omega_1, n) : n \in \mathbb{N}\} = \{\omega_1\} \times \omega_0$, an infinite sequence in level e with limit point (ω_1, ω_0) in the Tychonoff plank, $S \times \{e\}$. Let

$$R(e) = q[\{\omega_1\} \times \omega_0] \times \{e\}$$

In what follows suppose e represents a positive number greater than 0 and $p \in \mathbb{N} \setminus \{0\}$.

(a) If f maps a tail-end of $q[\{\omega_1\} \times \omega_0]$ at level e , into $\left[\frac{1}{p}, \infty\right)$ then f maps a tail-end of $q[\{\omega_1\} \times \omega_0]$ at level $e + 2$ into $\left[\frac{1}{p+1}, \infty\right)$

(b) If f maps a tail-end of $q[\{\omega_1\} \times \omega_0]$ at level $-e$, into $\left(-\infty, -\frac{1}{p}\right]$ then f maps a tail-end of $q[\{\omega_1\} \times \omega_0]$ at level $-e + 2$ into $\left(-\infty, -\frac{1}{p+1}\right]$

Proof of (a): We are given that the function $f : \mathcal{K} \rightarrow \mathbb{R}$ is continuous which maps a tail-end of $R(e)$ into the interval $\left[\frac{1}{p}, \infty\right)$.

Let $R(e)^*$ denote a tail-end of $R(e)$.

Observe that, at level e ,

$$\begin{aligned} f|_{T_{q(e)}}^\beta((\omega_1, \omega_0)) &\in f|_{T_{q(e)}}^\beta[\text{cl}_S R(e)^*] \\ &\subseteq \text{cl}_{\mathbb{R}}(f|_{T_{q(e)}}[R(e)^*]) \\ &\subseteq \text{cl}_{\mathbb{R}}\left[\frac{1}{p}, \infty\right) \quad (\text{By hypothesis.}) \\ &= \left[\frac{1}{p}, \infty\right) \end{aligned}$$

So, at level e ,¹

$$f|_{T_{q(e)}}^\beta((\omega_1, \omega_0)) \in \left[\frac{1}{p}, \infty\right) \quad (\dagger)$$

We are required to show that f maps a tail-end of the sequence $R(e - 2)$ (in level $e - 2$) into $\left[\frac{1}{p+1}, \infty\right)$.

Suppose not. Then $f^\leftarrow\left[\left(-\infty, \frac{1}{p+1}\right)\right]$ contains a subsequence, $R(e - 2)^{**}$, of $R(e - 2)$.

¹It is important to note that, by Theorem 21.18, $f|_{T_{q(e)}} : T_{q(e)} \rightarrow \mathbb{R}$ extends continuously to $f^\beta : [0, \omega_1] \times [0, \omega_0] \times \{e\} \rightarrow \mathbb{R}$.

Since $T_{q(e-2)} = T_{q(e-1)}$, then the set $f^{-1} \left[\left(-\infty, \frac{1}{p+1} \right) \right]$ contains a subsequence, $R(e-1)^{**}$, of $R(e-1)$.

Since f is constant on a tail-end of ω_1 , for each $n \in \mathbb{N}$, (see page 516) there is α_n such that f is constant on $(\alpha_n, \omega_1) \times \{n\}$. Let $\beta_0 = \sup\{\alpha_n : n \in \mathbb{N}\} < \omega_1$. (see page 107).

It follows that, in level $e-1$,

$$f[(\beta_0, \omega_1) \times R(e-1)^{**}] \subseteq \left(-\infty, \frac{1}{p+1} \right)$$

Let $S = ([0, \omega_1] \times [0, \omega_0]) \times \{e\}$.

$$\begin{aligned} f|_{T_{q(e-1)}}^\beta((\omega_1, \omega_0)) &\in f|_{T_{q(e-1)}}^\beta[\text{cl}_S((\beta_0, \omega_1) \times R(e-1)^{**})] \\ &\subseteq \text{cl}_{\mathbb{R}} f[(\beta_0, \omega_1) \times R(e-1)^{**}] \\ &\subseteq \text{cl}_{\mathbb{R}} \left(-\infty, \frac{1}{p+1} \right) \\ &= \left(-\infty, \frac{1}{p+1} \right] \end{aligned}$$

Then, in level $e-1$,

$$f|_{T_{q(e-1)}}^\beta((\omega_1, \omega_0)) \in \left(-\infty, \frac{1}{p+1} \right]$$

But, recall that in the top edge,

$$q[(\omega_1 \times \{\omega_0\}) \times \{e-1\}] = q[(\omega_1 \times \{\omega_0\}) \times \{e\}]$$

so

$$f|_{T_{q(e-1)}}^\beta((\omega_1, \omega_0)) = f|_{T_{q(e)}}^\beta((\omega_1, \omega_0))$$

Then

$$f|_{T_{q(e)}}^\beta((\omega_1, \omega_0)) \in \left(-\infty, \frac{1}{p+1} \right] \quad (\dagger\dagger)$$

Combining statement $(\dagger\dagger)$ with statement (\dagger) we obtain,

$$f|_{T_{q(e)}}^\beta((\omega_1, \omega_0)) \in \left(-\infty, \frac{1}{p+1} \right] \cap \left[\frac{1}{p}, \infty \right) = \emptyset$$

a contradiction.

The contradiction is the result of negating that, in level $e-2$, $\left[\frac{1}{p+1}, \infty \right)$ contains a tail-end of $R(e-2)$.

We conclude that $\left[\frac{1}{p+1}, \infty\right)$ must contain a tail-end of $R(e-2)$. This concludes the proof of part (a).

Proof of (b): For part (b) we proceed similarly.

We are given that e is an even integer < 0 and that the function $f : \mathcal{K} \rightarrow \mathbb{R}$ maps a tail-end of $R(-e)$ into the interval $\left(-\infty, -\frac{1}{p}\right]$. Let $R(-e)^*$ denote a tail-end of $R(-e)$.

As argued in part (a), at level $-e$,

$$f|_{T_{q(-e)}}^\beta((\omega_1, \omega_0)) \in \left(-\infty, -\frac{1}{p}\right] \quad (\dagger)$$

We are required to show that f maps a tail-end of the sequence $R(-e+2)$ (in level $-e+2$) into $\left(-\infty, -\frac{1}{p+1}\right]$.

Suppose not. Then $f^\leftarrow\left[\left(-\frac{1}{p+1}, \infty\right)\right]$ contains a subsequence, $R(-e+2)^{**}$, of $R(-e+2)$.

Since $T_{q(-e+2)} = T_{q(-e+1)}$, then the set $f^\leftarrow\left[\left(-\frac{1}{p+1}, \infty\right)\right]$ contains a subsequence, $R(-e+1)^{**}$, of $R(-e+1)$.

We then argue as in part (a) to conclude that at level $-e+1$,

$$f|_{T_{q(-e+1)}}^\beta((\omega_1, \omega_0)) \in \left[-\frac{1}{p+1}, \infty\right)$$

But, recall that in the top edge,

$$q[(\omega_1 \times \{\omega_0\}) \times \{-e+1\}] = q[(\omega_1 \times \{\omega_0\}) \times \{-e\}]$$

so

$$f|_{T_{q(-e+1)}}^\beta((\omega_1, \omega_0)) = f|_{T_{q(-e)}}^\beta((\omega_1, \omega_0))$$

Then

$$f|_{T_{q(-e)}}^\beta((\omega_1, \omega_0)) \in \left[-\frac{1}{p+1}, \infty\right) \quad (\dagger\dagger)$$

Combining statement $(\dagger\dagger)$ with previously established statement (\dagger) which states that $f|_{T_{q(-e)}}^\beta((\omega_1, \omega_0)) \in \left(-\infty, -\frac{1}{p}\right]$, we obtain,

$$f|_{T_{q(-e)}}^\beta((\omega_1, \omega_0)) \in \left(-\infty, -\frac{1}{p}\right] \cap \left[-\frac{1}{p+1}, \infty\right) = \emptyset$$

a contradiction.

The contradiction is the result of negating that, in level $-e+2$, $\left(-\infty, -\frac{1}{p+1}\right]$ contains a tail-end of $R(-e+2)$.

We conclude that, if $f^{-}\left[\left(-\infty, -\frac{1}{p}\right]\right]$ contains a tail-end of $R(-e)$ then $f^{-}\left[\left(-\infty, -\frac{1}{p+1}\right]\right]$ must contain a tail-end of $R(-e+2)$. This concludes the proof of part (b).

Theorem F3. Any continuous function, $f : \mathcal{X} \rightarrow \mathbb{R}$, on \mathcal{X} has the same sign at a^+ and a^- , hence f must map a^+ and a^- to the same number. Then \mathcal{X} is not completely regular.

Proof : We are given that the function $f : \mathcal{X} \rightarrow \mathbb{R}$ is continuous.

Claim #1. We claim that $f(a^+)$ and $f(a^-)$ must have the same sign.

Proof of claim: Suppose not. Say, for example, that $f(a^+) = r > 0$ and $f(a^-) = q < 0$.

Let $p \in \mathbb{N} \setminus \{0\}$ such that $1/p < \min\{r, |q|\}$. Then $[r, \infty) \subseteq (1/p, \infty)$ and $(-\infty, q] \subseteq (-\infty, -1/p)$.

Then

$$\begin{aligned} a^+ \in f^{-}[[r, \infty)] &\subseteq f^{-}[(1/p, \infty)] \\ a^- \in f^{-}[(-\infty, q]] &\subseteq f^{-}[(-\infty, -1/p)] \end{aligned}$$

Then $f^{-}[(1/p, \infty)]$ and $f^{-}[(-\infty, -1/p)]$ are disjoint open neighborhoods of a^+ and a^- in \mathcal{X} , respectively.

There exists an even number e in $\mathbb{N} \setminus \{0\}$ such that,

$$\begin{aligned} B_e(a^+) &= \{a^+\} \cup \{T_{q(e)}, T_{q(e+1)}, T_{q(e+2)}, \dots\} \subseteq f^{-}[(1/p, \infty)] \\ B_{(-e)}(a^-) &= \{a^-\} \cup \{T_{q(-e)}, T_{q(-e-1)}, \dots\} \subseteq f^{-}[(-\infty, -1/p)] \end{aligned}$$

Then

$$\begin{aligned} f[B_e(a^+)] &\subseteq f[f^{-}[(1/p, \infty)]] = (1/p, \infty) \\ f[B_{(-e)}(a^-)] &\subseteq f[f^{-}[(-\infty, -1/p)]] = (-\infty, -1/p) \end{aligned}$$

Recall that

$$R(e) = q[\{\{\omega_1\} \times \omega_0\} \times \{e\}] \subseteq T_{q(e)}$$

Then f maps a tail-end of $R(e)$ into $f[T_{q(e)}]$.

Since,

$$\begin{aligned} f[T_{q(e)}] &\subseteq f[B_e(a^+)] \\ &\subseteq (1/p, \infty) \end{aligned}$$

then, f maps a tail-end of $R(e)$ into $[1/p, \infty)$.

By the same reasoning, f maps a tail-end of $R(-e)$ into $(-\infty, -1/p]$.

We now show how this will lead to a contradiction.

By Lemma F2, f maps a tail-end of $R(e-2)$ into $[\frac{1}{p+1}, \infty)$, and f maps a tail-end of $R(-e+2)$ into $(-\infty, -\frac{1}{p+1}]$.

To see how this will lead to a contradiction, suppose $e = 6$, and let $R(6)^*$ and $R(-6)^*$ denote tail-ends of $R(6)$ and $R(-6)$.

$$\begin{aligned} f[R(6)^*] \subseteq \left[\frac{1}{p}, \infty\right) &\Rightarrow f[R(4)^*] \subseteq \left[\frac{1}{p+1}, \infty\right) \\ &\Rightarrow f[R(2)^*] \subseteq \left[\frac{1}{p+2}, \infty\right) \\ &\Rightarrow f[R(0)^*] \subseteq \left[\frac{1}{p+3}, \infty\right) \end{aligned}$$

$$\begin{aligned} f[R(-6)^*] \subseteq \left(-\infty, -\frac{1}{p}\right] &\Rightarrow f[R(-4)^*] \subseteq \left(-\infty, -\frac{1}{p+1}\right] \\ &\Rightarrow f[R(-2)^*] \subseteq \left(-\infty, -\frac{1}{p+2}\right] \\ &\Rightarrow f[R(0)^*] \subseteq \left(-\infty, -\frac{1}{p+3}\right] \end{aligned}$$

We then obtain the contradiction

$$f[R(0)^*] \subseteq \left(-\infty, -\frac{1}{p+3}\right] \cap \left[\frac{1}{p+3}, \infty\right) = \emptyset$$

The source of the contradiction is the supposition that $f(a^+)$ is positive and $f(a^-)$ is negative. We conclude that for any continuous function $f : \mathcal{K} \rightarrow \mathbb{R}$, $f(a^+)$ and $f(a^-)$ must have the same sign. This completes the proof of the Claim #1,

Claim #2: We claim that for any continuous function $f : \mathcal{K} \rightarrow \mathbb{R}$, $f(a^+)$ and $f(a^-)$ cannot have different values.

To see this, suppose $f(a^+) = u$ and $f(a^-) = v$ where $0 \leq u < v$. Let $z \in \mathbb{R}$ such that $u < z < v$ and $g(x) = x - z$. Then $g \circ f : \mathcal{X} \rightarrow \mathbb{R}$ is a continuous function such that $(g \circ f)(a^+)$ and $(g \circ f)(a^-)$ have a different sign. This has been shown to be impossible.

We conclude that there can be no real-valued continuous function $f : \mathcal{X} \rightarrow \mathbb{R}$ which can separate a^+ and a^- .

Then \mathcal{X} is not completely regular.