3 / A topology on a set

Abstract. *In this section we define the notion of a topological space. We will begin by describing those families of subsets of a set which form a topology, τ, on a set, showing along the way how to recognize "open subsets" and "closed subsets". There can be many topologies on a set. Given a pair of topologies τ_1 and τ_2 on S one can sometimes be seen as being "weaker" or "stronger" than the other. We have provided many examples both in the main body of the text as well as in the given exercises.*

3.1 Introduction.

In our review of metric spaces we realized, in Theorem 2.10, that a function, $f : S \to T$, from a metric space S to another metric space T is *continuous* whenever the following condition is satisfied: "The set, $f^{\leftarrow}[U]$, is open in S whenever the set, U, is open in T". We learned that, some sets are described as being "open", others as being "closed", some are both open and closed, while some are neither. We then learned how to distinguish one from the other. For example, we recognized closed sets as being those whose complement is open. We can recognize a closed set, F, as being one that "contains the limit point of every convergent sequence in F". It seems that knowing various characterizations for each type of set is not only useful, but important.

How did this notion of an "open set" originate? Recall that we constructed open sets in a set S with the help of a norm or metric previously defined on S. These two distance measuring tools defined on S (themselves inspired by the notions of "absolute value" and "open interval" in $\mathbb{R}$) lead us to define the concept of an "open ball" in S which, in turn, allowed us to recognize those subsets, T, of S which are open.

We would now like to define "open set" in a much more abstract context, in a way that is independent of any distance measuring tool such as a norm or a metric. Spaces whose properties will depend specifically on how we define its "open sets" will be called *topological spaces.* The family of "open sets" in a set must however satisfy basic open set axioms or rules.

Definition 3.1 Let S be a non-empty set.

(a) A *topology* on the set S is a collection, τ, of subsets of S which possesses the following properties:

O1. Both the empty set, $\varnothing$, and S belong to τ.

O2. If $\mathscr{C} \subseteq \tau$, then $\cup\{C \in \mathscr{C}\}$ is a set which also belongs to τ.

O3. If $\mathscr{F}$ is a finite subset of τ, then $\cap\{C \in \mathscr{F}\}$ is also a set which belongs to τ.

We will refer to O1, O2 and O3 as the *open set axioms*.

(b) If τ is a topology on a set S (as defined above), then each member, $U \in \tau$, is called an

open subset of S

If $U \in \tau$, we often simply say that "U is *open* in S".

(c) Suppose we have defined a topology, τ, on some set S. Then this set, S, when considered together with this topology τ, is called a *topological space* and can be represented as (S, τ). The condition, O2, can also be expressed by the phrase "*τ is closed under arbitrary unions*". The condition, O3, can also be expressed by the phrase "*τ is closed under finite intersections*".

It is important to remember that the elements of a topology τ on S are all subsets of S. When we say that S is an open subset of S, it is assumed that a topology τ has been previously defined; that is, that τ satisfies the three properties stated above. There can be many topologies defined on a given set. If τ_1 and τ_2 are two different topologies on a set S, then (S, τ_1) and (S, τ_2) are considered to be distinct topological spaces even though they contain the same points. When we view (S, τ_1) and (S, τ_2) simply as *sets* they are of course equal. We speak of U as being an open subset of S when it is clearly understood from the context that U is a member of a predefined topology τ on S.

Example 1. Suppose (S, ρ) is a metric space and $B_\varepsilon(y) = \{x \in S : \rho(x, y) < \varepsilon\}$ represents an open ball of radius ε centered at y in S. Let

$$\tau_\rho = \{\varnothing\} \cup \{U \subseteq S : U \text{ is the union of open balls in } S\}$$

Then τ_ρ is a topology on S. (Showing that τ_ρ satisfies the three open set axioms is left as an exercise.) We say that τ_ρ is the

topology induced by the metric ρ

on S. If S is vector space on which we have defined a norm, we sometimes specify this by referring to S as a

normed topological space

and we might write $\tau_{\| \ \|}$.

> If a metric, ρ, is defined on S, a set A belongs to τ_ρ if and only if A is open in the metric space (S, ρ). In this sense, every metric space can be viewed as a topological space.

Note : There exist topological spaces whose topology is not derived from a metric ρ. Those topological spaces whose open sets can be derived from some metric have a special name.

Definition 3.2 Let (S, τ) be a topological space. We will say that (S, τ) is *metrizable* if there exists a metric, ρ, on S such that $\tau = \tau_\rho$ (where τ_ρ is induced on S by ρ).

From this definition, we can state that the class of all topological spaces can be subdivided into two subfamilies: One consists of all metrizable spaces the other of non-metrizable ones.

For example, the usual metric $\rho(x, y) = |x-y|$ on $\mathbb{R}$ induces a topology, τ_ρ, on $\mathbb{R}$. We normally refer to this topology as being the *usual topology on* $\mathbb{R}$. In many textbooks, it is also referred to as the *Euclidean topology* on $\mathbb{R}$. In this case, the open subsets of $\mathbb{R}$ are the sets which are unions of open intervals $B_\varepsilon(y) = (y - \varepsilon, y + \varepsilon)$ and the empty set $\varnothing$. For example, the sets $U_1 = \cup\{(n, n+1) : n \in \mathbb{Z}\}$ and $U_2 = (-4, 9)$ can be shown to belong to τ_ρ, but the subset $U_3 = [-7, -3) \notin \tau_\rho$. (It is left to the reader to verify this.)

Example 2. *A non-metrizable space.* Consider the set $S = \mathbb{R}^2$ equipped with the topology

$$\tau = \{B \subseteq \mathbb{R}^2 : \mathbb{R}^2 \backslash B \text{ is countable}\}^1 \cup \{\varnothing, \mathbb{R}^2\}$$

(a) Verify that the family, τ, of subsets of $\mathbb{R}^2$ is indeed a topology on $\mathbb{R}^2$.

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[1] If A is a subset of a space S, then $S \backslash A$ is the complement of A. It denotes the set of all points in S which don't belong to A.

(b) Verify that the topological space, $\mathbb{R}^2$, equipped with the topology, τ, described above is *not* metrizable.

Solution : Given: $\tau = \{B \subseteq \mathbb{R}^2 : \mathbb{R}^2 \backslash B \text{ is countable}\} \cup \{\varnothing, \mathbb{R}^2\}$.

(a) Verification that the sets in τ satisfy the three open sets axioms O1, O2, and O3 is left as an exercise.

(b) Suppose (S, τ) is metrizable. That is, suppose ρ is a metric on $\mathbb{R}^2$ such that $\tau = \tau_\rho$. We will show that, given $\tau = \tau_\rho$, then $\rho(\vec{x}, \vec{y}) > \rho(\vec{x}, \vec{z}) + \rho(\vec{y}, \vec{z})$, contradicting the fact that ρ is a valid metric on $\mathbb{R}^2$.

Suppose $\vec{x}$ and $\vec{y}$ are distinct points in $\mathbb{R}^2$ and $\rho(\vec{x}, \vec{y}) = \alpha \neq 0$. Consider the open balls $B_{\alpha/2}(\vec{x})$ and $B_{\alpha/2}(\vec{y})$. Since both balls belong to τ, each ball has a countable complement; so each ball is an uncountable subset. Hence $B_{\alpha/2}(\vec{y})$ cannot be entirely contained in $\mathbb{R}^2 \backslash B_{\alpha/2}(\vec{x})$. That means that $B_{\alpha/2}(\vec{y}) \cap B_{\alpha/2}(\vec{x}) \neq \varnothing$. Let $\vec{z} \in B_{\alpha/2}(\vec{y}) \cap B_{\alpha/2}(\vec{x})$. Then

$$\begin{aligned} \rho(\vec{x}, \vec{z}) + \rho(\vec{y}, \vec{z}) &< \alpha/2 + \alpha/2 \\ &= \alpha \\ &= \rho(\vec{x}, \vec{y}) \end{aligned}$$

We see that $\rho(\vec{x}, \vec{y}) > \rho(\vec{x}, \vec{z}) + \rho(\vec{y}, \vec{z})$ and so ρ does not satisfy the triangle inequality, contradicting the fact that ρ is a metric on $\mathbb{R}^2$. So there can be no metric ρ on $\mathbb{R}^2$ such that $\tau = \tau_\rho$. We conclude that the topological space, $(\mathbb{R}^2, \tau)$, is not a metrizable space.[2]

If U is open and T is a subset of S such that $x \in U \subset T$, the subset, T, is said to be a *neighborhood of* x. We will more formally define this in the next chapter.

Note that a point, x, in a topological space, (S, τ), always has at least one open neighborhood, namely S. If $T = (-2, 4) \cup (4, 7]$ in $\mathbb{R}$, equipped with the usual topology, we see that T is a neighborhood of 3 (since $3 \in (\frac{5}{2}, \frac{7}{2}) \subseteq T$) but not a neighborhood of 4 nor of 7 (since there is no open set U such that $7 \in U \subseteq T$).

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[2] Later, once we have covered the concept of "Hausdorff" this problem will be more easily solved by stating that "this space is not metrizable because it not Hausdorff".

Definition 3.3 Suppose τ_1 and τ_2 are two topologies on a given set S. If $\tau_1 \subseteq \tau_2$, then we say that

$$\tau_1 \textit{ is a weaker topology than } \tau_2$$

on S or that

$$\tau_2 \text{ is a } \textit{stronger topology than } \tau_1$$

on S. We can also say that τ_1 is a *coarser topology than* τ_2 on S, or that τ_2 is a *finer topology than* τ_1 on S. We will say that the two topological spaces (S, τ_1) and (S, τ_2) are *equivalent* if and only if $\tau_1 = \tau_2$.

Example 3. Suppose S is a non-empty set and $\mathscr{P}(S)$ denotes the power set of S (that is, $\mathscr{P}(S)$ denotes the collection of all subsets of S). Let $\tau_d = \mathscr{P}(S)$. Then τ_d is a topology on S. It is left to the reader to verify this. In this case, for every single point $x \in S$, $\{x\}$ is open in S.

For a given space S, if

$$\tau_d = \mathscr{P}(S)$$

the topology, τ_d, is referred to as the

$$\textit{discrete topology on } S$$

If S is equipped with the discrete topology, then every non-empty subset of S is an open neighborhood of the elements it contains.

Example 4. For a non-empty set S, if $\tau_i = \{\varnothing, S\}$, then the two-element set, τ_i, is a topology on S.

If $\tau_i = \{\varnothing, S\}$, the topology, τ_i, is normally referred to as the

$$\textit{indiscrete topology on } S$$

In this case, for any $x \in S$, S is its only neighborhood.

Since all topologies on S must at least contain $\varnothing$ and S, τ_i is the weakest (coarsest) of all topologies on S. On the other hand, $\tau_d = \mathscr{P}(S)$ is the strongest (finest) of all topologies on S. For any topology τ on S, we then have

$$\tau_i \subseteq \tau \subseteq \tau_d$$

Note on the intersection of topologies: Given a family $\{\tau_k : k \in I\}$ of topologies on a set S, it is an easy exercise to show that the family $\cap_{k \in I} \tau_k$ is also a topology on S. It is left to the reader to verify that this is so. This fact can also be expressed as: "*The set of all topologies is closed under arbitrary intersections.*"[3]

3.2 Closed subsets of a topological space.

In our brief overview of metric spaces, (S, ρ), we defined a closed subset, F, of S as one which contains all its limit points. We then saw that, in a metric space S, a subset, F, of S is closed if and only if its complement, $S \backslash F$, is open.[4] Metric spaces were equipped with a "metric" so that we could discuss the important notions of "convergence of a sequence" and "closed set" in a set S. We now formally define the analogous notions of "closed subsets in a topological space". We will not discuss the concepts of "convergence" and "limit points" in a topological space immediately. We will be doing an in-depth study of this topic in a section later in this book.

Definition 3.4 Let F be a subset of a topological space (S, τ). If the complement, $S \backslash F$, of F is an open subset in S, then we say that F is a *closed subset in* S.

The definition of "closed" states that "($S \backslash F$ is open) $\Rightarrow$ (F is closed")". Conversely,

$$\begin{aligned} A \text{ is closed} &\Rightarrow A = S \setminus [S \backslash A] \text{ is closed} \\ &\Rightarrow [S \backslash A] \text{ is open} \end{aligned}$$

We can then actually write

$$(F \text{ is closed}) \Leftrightarrow (S \backslash F \text{ is open})$$

Suppose (S, τ) is a topological space and $\mathscr{F} = \{A \subset S : A \text{ is closed in } S\}$. Then we can define the topology in terms of $\mathscr{F}$ as follows:

$$\tau = \{A : S \backslash A \in \mathscr{F}\}$$

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[3] Caution: However, arbitrary unions of topologies may not form a topology on a set. For example, verify that $\tau_1 = \{\varnothing, S, \{b\}, \{a, b\}\}$ and $\tau_2 = \{\varnothing, S, \{c\}, \{a, c\}\}$ are two topologies on S; but their union, $\tau_1 \cup \tau_2$, is not a topology on S. (Since $\{a, b, c\} \notin \tau_1 \cup \tau_2$. Verify this.)

[4] $S \backslash F = \{x \in S : x \notin F\}$

Theorem 3.5 Let (S, τ) be a topological space and I be any indexing set. Then,

(a) Both $\varnothing$ and S are closed in S.

(b) If $\{F_i : i \in I\}$ is a family of closed subsets of S, then $\cap\{F_i : i \in I\}$ is a closed subset of S.

(c) If $\{F_i : i = 1, 2, 3, \ldots, k\}$ is a finite family of closed subsets of S, then $\cup\{F_i : i = 1, 2, 3, \ldots, k\}$ is a closed subset of S.

Proof:

a) Since $\varnothing$ is open, then $S = S\backslash\varnothing$ is closed. Since S is open, $\varnothing = S\backslash S$ is closed.

b) Let $\{F_i : i \in I\}$ be a family of closed subsets of S. Then, for each $i \in I$, $S\backslash F_i$ is open. Since (by De Morgan's law)

$$S\backslash\cap\{F_i : i \in I\} = \cup\{S\backslash F_i : i \in I\}$$

is open (being the union of open sets), then $\cap\{F_i : i \in I\}$ is a closed subset of S.

c) This part is left as an exercise.

Defining a topology on S in terms of "closed sets". Suppose we are given a set S and family, $\mathscr{F} = \{F : F \subseteq S\}$ of elements from $\mathscr{P}(S)$ which satisfies the following three conditions:

F1. The sets $\varnothing$, S both belong to $\mathscr{F}$.
F2. If $\{F_i : i \in I\} \subseteq \mathscr{F}$, then $\cap\{F_i : i \in I\} \in \mathscr{F}$.
F3. If $\{F_i : i = 1, 2, 3, \ldots, k\}$ is a finite subset of $\mathscr{F}$, then $\cup\{F_i : i = 1, 2, 3, \ldots k\} \in \mathscr{F}$.

Then the family, $\tau = \{S\backslash F : F \in \mathscr{F}\}$, forms a topology on S.

The proof showing that τ is a topology is left to the reader. It easily follows from an application of De Morgan's law[5].

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[5] That is, $S \setminus [\cup_{i\in I} F_i] = \cap_{i\in I}[S\backslash F_i]$ and $S \setminus [\cap_{i\in I} F_i] = \cup_{i\in I}[S\backslash F_i]$

The conditions F1, F2 and F3 described above are referred to as the...

“*closed set axioms*”

Example 5. Suppose S is a non-empty set and

$$\mathscr{F} = \{F : F \text{ is a finite subset of } S\} \cup \{\varnothing, S\}$$

Then the set $\mathscr{F}$ satisfies the three closed sets axioms F1, F2 and F3. (Verify this!) Then $\tau = \{S\backslash F : F \in \mathscr{F}\}$ forms a topology on S. The elements of τ are

$$\tau = \{\varnothing, S\} \cup \{U \subset S : S\backslash U \text{ is finite }\}$$

Given a set S, the family of subsets,

$$\tau = \{A : A \subseteq S \text{ and } S\backslash A \text{ is finite }\} \cup \{S, \varnothing\}^6$$

forms a topology of S called the *cofinite topology on* S or the *Zariski topology on* S.

3.3 Subspace topology on a subset.

We previously defined the notion of a “metric subspace”, T, as being a subset of a metric space, (S, ρ), equipped with the subspace metric, ρ_T.

In the more general case of a topological space, (S, τ), any subset T can be declared to be a “topological subspace” provided the reader understands what topology is defined on T. Suppose H is a non-empty subset of a topological space (S, τ). Then H can inherit its topology from τ, in a natural way, as shown the following theorem.

Theorem 3.6 Let (S, τ) be topological space and $H \subseteq S$.

(a) Let

$$\tau_H = \{U \subset H : U = K \cap H, \text{ for some } K \in \tau\}$$

Then τ_H is a topology on H.

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[6] If $S\backslash U$ is finite we say U is cofinite.

(b) Let

$$\mathscr{F}_H = \{F \subset H : F = M \cap H \text{ where } M = S \setminus K, K \in \tau\}$$

Then $\mathscr{F}_H$ represents *all* closed subsets of the topological space (H, τ_H)

Proof: The proof is left as an exception.

Definition 3.7 If (S, τ) is topological space and $H \subseteq S$ and

$$\tau_H = \{U \subset H : U = K \cap H,\ K \in \tau\}$$

then τ_H is called the

subspace topology

or *relative topology* on H induced by S, or inherited from S. In such a case, we will say that (H, τ_H) is a *subspace of S*.

Some subsets of a topological space (S, τ) can be both open and closed. For example, if $(\mathbb{R}, \tau_i)$ is equipped with the indiscrete topology, $\tau_i = \{\varnothing, \mathbb{R}\}$, both $\varnothing$ and $\mathbb{R}$ are the only open subsets of $\mathbb{R}$ and so both $\varnothing$ and $\mathbb{R}$ are the only closed subsets of $\mathbb{R}$. That is, $\varnothing$ and $\mathbb{R}$ are simultaneously open and closed in $\mathbb{R}$. We consider a less trivial example.

Example 6. Let $H = [3, 5)$. Consider the subset

$$T = H \cup \{9\} = [3, 5) \cup \{9\}$$

of $(\mathbb{R}, \tau)$ where $\mathbb{R}$ is equipped with the usual topology, τ. Suppose the subspace (T, τ_T) is equipped with the subspace topology inherited from τ. Now H is a subset of both T and $\mathbb{R}$. Since $H = T \cap [2, 6]$, then H is closed in T with respect to the subspace topology τ_T. Since $H = T \cap (2, 5)$, then H is open in T with respect to the subspace topology τ_T. So the subset, H, is both open and closed in the subspace, T. Note, however, that H is neither open nor closed in S.

A subset T of a topological space (S, τ), which is both open and closed with respect to the topology τ, is commonly said to be

"*clopen in S*."

For example, let $(\mathbb{Q}, \tau_{\mathbb{Q}})$ be the subspace inherited from $(\mathbb{R}, \tau)$ equipped with the usual topology. Then the subset, $A = (\sqrt{2}, \pi) \cap \mathbb{Q}$, is clopen in $(\mathbb{Q}, \tau_{\mathbb{Q}})$, since both A and $\mathbb{Q}\backslash A$ are open in $(\mathbb{Q}, \tau_{\mathbb{Q}})$.

Question: Are the points of the topological space $(\mathbb{Q}, \tau_{\mathbb{Q}})$ clopen subsets?

Note: Suppose (S, τ) is a topological space which contains the non-empty subset, (F, τ_F), equipped with the subspace topology, τ_F, inherited from S. If one speaks of an open subset, U, of F, it may not be obvious to the reader whether $U \in \tau$ or $U \in \tau_F$. If $U \in \tau$, it helps to be more specific by writing

"U is an S-open subset in F"

and, if $U \in \tau_F$,

"U is an F-open subset of F"

Question: What if F is open in S, and U is open in F?[7]

3.4 Other examples.

We provide a few more examples of topological spaces.

Example 7. Let S be a set and $B \subseteq S$. Let $\tau_B = \{A \in \mathscr{P}(S) : B \subseteq A\} \cup \{\varnothing\}$. That is, the only open subsets of S are those that contain the set B or one which is the empty set.

(a) Verify that τ_B is indeed a topology on S.

(b) Describe the closed subsets of (S, τ_B).

Solution: We are given that $B \subseteq S$ and $\tau_B = \{A \in \mathscr{P}(S) : B \subseteq A\} \cup \{\varnothing\}$.

(a) We verify that τ_B is a topology on S by confirming that it satisfies the open set axioms O1, O2 and O3.
   - By definition, $\varnothing \in \tau_B$; also, since $B \subseteq S$, then $S \in \tau_B$.
   - Suppose $\mathscr{U}$ is a non-empty subset of τ. Then $B \subseteq U$ for each $U \in \mathscr{U}$. So $B \subseteq \cup\{U : U \in \mathscr{U}\}$. Hence $\cup\{U : U \in \mathscr{U}\} \in \tau_B$.
   - Suppose $\mathscr{F}$ is a finite subset of τ_B. Then $B \subseteq F$ for each $F \in \mathscr{F}$. So $B \subseteq \cap\{F : F \in \mathscr{F}\}$. Hence $\cap\{F : F \in \mathscr{F}\} \in \tau_B$.

(b) We now describe the closed subsets of S.

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[7] If F is S-open and U is an F-open subset of F, then U is also S-open. Verify this!

Suppose $A \subseteq S$. We wish to determine when A is a closed subset of S. Either $A \cap B = \varnothing$ or $A \cap B \neq \varnothing$.

Case 1. If $A \cap B = \varnothing$, then $B \subseteq S \setminus A$; this means $S \setminus A$ is an open subset of S. We conclude that whenever $A \cap B = \varnothing$, A is a closed subset of S.

Case 2. Suppose $A \cap B \neq \varnothing$. We claim that, if $A \cap B \neq \varnothing$, then S is the only closed subset which can contain A.
Proof of claim: Suppose that F is any closed subset of S which contains A. Then $S \backslash F$ is an open subset which contains B or $S \backslash F = \varnothing$. If $B \subseteq S \backslash F$ then $B \cap F = \varnothing$. But $A \cap B \subseteq A \subseteq F$. Since $A \cap B \neq \varnothing$ this is impossible. So $S \backslash F$ must be $\varnothing$. So the closed subset F can only be S.

So the closed subsets of S are the family

$$\mathscr{F} = \{F \subset S : F \cap B = \varnothing\} \cup \{S\}$$

Remark. *Note that the singleton sets of a topological space need not necessarily be closed subsets.* For example suppose that in the previous example, $B = \{a, b\}$. Since $\{a\} \cap B = \{a\} \neq \varnothing$ and $\{a\} \neq S$ then $\{a\}$ is not a closed subset of S. Furthermore, since $B \not\subseteq \{a\}$ then $\{a\}$ is not an open subset of S.

Example 8. Let (S, τ) be a topological space and suppose B is a fixed subset of S. Let

$$\tau_B = \{A \in \mathscr{P}(S) : A = C \cup (D \cap B) \text{ where } C, D \in \tau\}$$

(a) Verify that τ_B is another topology on S.

(b) Is one of the two topologies, τ_B, τ, stronger than the other? Are these two topologies equivalent topologies?

Solution: We are given that (S, τ) is a topological space and B is a fixed subset of S.

(a) We begin by showing that τ_B satisfies the three open set axioms O1, O2 and O3.
   - We know $\varnothing, S \in \tau$. So we have $\varnothing = \varnothing \cup (\varnothing \cap B) \in \tau_B$ and $S = S \cup (S \cap B) \in \tau_B$.
   - Suppose $\mathscr{U} = \{C_i \cup (D_i \cap B) : i \in I\} \subseteq \tau_B$, where $C_i, D_i \in \tau$ for all $i \in I$. Then $\cup_{i \in I} C_i$, and $\cup_{i \in I} D_i$ both belong to τ. Then

$$\bigcup_{i \in I} [C_i \cup (D_i \cap B)] = [\cup_{i \in I} C_i] \ \cup \ ([\cup_{i \in I} D_i] \cap B) \in \tau_B$$

   Thus τ_B is closed under arbitrary unions.

– Suppose $\mathscr{F} = \{C_i \cup (D_i \cap B) : i = 1, 2, \ldots, n\} \subseteq \tau_B$. Then

$$\begin{aligned}
\bigcap_{i=1,\ldots,n} [C_i \cup (D_i \cap B)] &= \bigcap_{i=1,\ldots,n} [(C_i \cup D_i) \cap (C_i \cup B)] \\
&= [\cap_{i=1,\ldots,n}(C_i \cup D_i)] \\
&\qquad \cap [(\cap_{i=1,\ldots,n} C_i) \cup B] \\
&= [\cap_{i=1,\ldots,n}(C_i \cup D_i)] \cap [(\cap_{i=1,\ldots,n} C_i)] \\
&\qquad \cup \ [\cap_{i=1,\ldots,n}(C_i \cup D_i) \cap B] \\
&\in \tau_B
\end{aligned}$$

So τ_B satisfies the three open set axioms O1, O2 and O3.

(b) We claim that $\tau \subseteq \tau_B$: See that, if $C \in \tau$, then $C = C \cup (\varnothing \cap B) \in \tau_B$, so $\tau \subseteq \tau_B$. So τ_B is a topology on S which is *finer* (*stronger*) than τ.

We claim that these two topologies are not equivalent: Suppose $B \notin \tau$, $D \in \tau$ and $B \subset D$. Then $\varnothing \cup (D \cap B) = B \notin \tau$. Since $B \in \tau_B$, then τ_B contains elements which are not in τ. So $\tau \subset \tau_B$.

In the above example, we say that "the topology τ_B *extends* τ *over* B".

3.5 Topics: Other distinguished topological subsets.

Besides the fundamental *open* and *closed* subsets of a topological space introduced earlier, there are other subsets defined in terms of open and closed sets with special properties. Two of these are called G-delta (G_δ) sets and F-sigma (F_σ) sets. It will be good practice to develop some familiarity with these now. As well, it allows us to freely refer to these in various examples, theorems and exercise questions further on.

The G-delta and F-sigma sets in a topological space. We have seen that arbitrary unions of open subsets of a topological space are open. However, only the intersection of (at most) finitely many open sets are guaranteed to be open. Similarly, arbitrary intersections of closed sets are closed, but the union of (at most) finitely many closed sets are guaranteed to be closed. The intersection of countably many open sets and the union of countably many closed sets may not, respectively, be open and closed, but such sets may still be relevant in our study of various types of spaces.

Before we continue, we remind the reader that a non-empty set S is said to be "countable" if it is finite or, in the case where it is infinite, the elements of S can be indexed by the natural numbers. That is,

$S = \{x_i : i = 0, 1, 2, 3, \ldots\}$. We can also say that the infinite set S is countable if there exists a function $f : \mathbb{N} \to S$ mapping the natural numbers *onto* S. We now formally define those special subsets of a topological space we call *G-delta*'s and *F-sigma*'s.

Definition 3.8 The sets in a topological space, (S, τ), which are the intersection of at most countably many open sets are called G_δ-sets (or simply G_δ). Those sets in S which are the union of at most countably many closed sets are called F_σ-sets (or simply F_σ). Neither of these special sets need be open or closed, respectively.

Trivially, if F is closed in (S, τ), then F is an F_σ and, if U is open, then U is a G_δ.

Example 9. If $\mathbb{R}$ is equipped with the usual topology, the set $T = [2, 7]$ is obviously an F_σ. It is also a G_δ since

$$[2, 7] \quad = \quad \cap\{(2 - 1/n, 7 + 1/n) : n = 1, 2, 3, \ldots\}$$

So some sets can be both a G_δ and an F_σ with respect to the same topology τ. The set $(2, 7]$ equal to $\cap\{(2, 7 + 1/n) : n = 1, 2, 3, \ldots\}$ is a G_δ. Since $(2, 7] = \cup\{[2 + 1/n, 7] : n = 1, 2, 3, \ldots\}$ it is also an F_σ.

Example 10. On the other hand, suppose (S, τ_i) is equipped with the indiscrete topology, $\tau_i = \{\varnothing, S\}$. If T is a proper non-empty subset of S, we see that $T \nsubseteq \varnothing$ and $T \neq S$; so the proper subset, T, is neither a G_δ nor an F_σ with respect to τ_i.

Example 11. We consider the set of all rationals, $\mathbb{Q}$, as a subset of $\mathbb{R}$ equipped with the usual topology. It is known that $\mathbb{Q}$ is countably infinite and so can be expressed in the form $\mathbb{Q} = \{x_i : i = 1, 2, 3, \ldots\}$. Then $\mathbb{Q} = \cup\{\{x_i\} : i = 1, 2, 3, \ldots\}$ where each $\{x_i\}$ is a closed subset of $\mathbb{R}$. So $\mathbb{Q}$, when viewed as a proper subset of $\mathbb{R}$, is an F_σ.[8]

The following theorem exhibits properties respected by each F_σ and G_δ and the families of all G_δ's and F_σ's of a topological space.

---

[8] Is $\mathbb{Q}$ a G_δ? Once we have the necessary tools, a bit later on, we will prove that $\mathbb{Q}$ is not a G_δ

Theorem 3.9 Suppose F is an F_σ and G is a G_δ in S.

(a) The complement of F in S is a G_δ and the complement of G in S is an F_σ.

(b) There exists a sequence, $\{F_i : i = 1, 2, 3, \ldots\}$, of closed subsets of S such that

$$F_i \subseteq F_{i+1} \text{ for all } i = 1, 2, 3, \ldots, \text{ and } F = \cup\{F_i : i = 1, 2, 3, \ldots\}$$

(c) There exists a nonincreasing sequence, $\{G_i : i = 1, 2, 3, \ldots\}$, of open subsets of S such that

$$G_{i+1} \subseteq G_i \text{ for all } i = 1, 2, 3, \ldots, \text{ and } G = \cap\{G_i : i = 1, 2, 3, \ldots\}$$

(d) Suppose $\mathscr{F}$ denotes the family of all F_σ's in S and $\{F_i : i = 1, 2, 3, \ldots\}$ represents at most countably many elements in $\mathscr{F}$. Then

$$\cup\{F_i : i = 1, 2, 3, \ldots\} \in \mathscr{F} \text{ and } \cap\{F_i : i = 1, 2, 3, \cdots, k\} \in \mathscr{F} \text{ for any } k$$

(e) Suppose $\mathscr{G}$ denotes the family of all G_δ's in S and $\{G_i : i = 1, 2, 3, \ldots\}$ represents at most countably many elements in $\mathscr{G}$. Then

$$\cap\{G_i : i = 1, 2, 3, \ldots\} \in \mathscr{G} \text{ and } \cup\{G_i : i = 1, 2, 3, \cdots, k\} \in \mathscr{G} \text{ for any } k$$

Proof: Given: (S, τ) be topological space; F is an F_σ and G is a G_δ in S.

(a) Suppose $F = \cup\{K_i : i = 1, 2, 3, \ldots,$ where each K_i is closed in $S\}$

$$\begin{aligned} S\backslash F &= S\backslash(\cup\{K_i\}) \\ &= \cap\{S\backslash K_i\} \quad \text{(By De Morgan's rule)} \\ &= \text{ a } G_\delta\text{-set} \end{aligned}$$

The proof of the second part of (a) follows by a similar application of De Morgan's rule.

(b) The proof is left as an exercise for the reader.

(c) The proof is left as an exercise for the reader.

(d) For countable unions of F_σ's:

$$\bigcup_{j=1}^{\infty}[\cup\{F_{(i,j)} : i = 1, 2, 3, \ldots, \}] = \cup\{F_{(i,j)} : (i, j) \in \mathbb{N} \times \mathbb{N}\}$$

where $\mathbb{N} \times \mathbb{N}$ is known to be countable.

For finite intersections of F_σ's:

$$\bigcap_{i=1}^{k} [\cup_{j=1}^{\infty} \{F_{(i,j)}\}] = \bigcup_{(j_1, \cdots, j_k) \in \mathbb{N} \times \mathbb{N} \times \cdots \times \mathbb{N}} \{F_{(1,j_1)} \cap \cdots \cap F_{(k,j_k)}\}$$

where $\mathbb{N} \times \mathbb{N} \times \cdots \times \mathbb{N}$ is known to be countable.

(e) This part is proved similarly to part (d).

We summarize two of the results in the above theorem:

> "The family, $\mathscr{F}$, of all F_σ's of a topological space is closed under countable unions and closed under *finite* intersections."
>
> "The family, $\mathscr{G}$, of all G_δ's of a topological space is closed under countable intersections and closed under *finite* unions."

We will see a bit later that, in certain specific classes of topological spaces, F_σ's are closed and G_δ's are open. We have to do some groundwork before we can discuss these.

3.6 Topics: Another distinguished family of subsets.[9]

Topological spaces each contain a particular family of subsets of $\mathscr{P}(S)$ which play a role in certain fields of study where topology is applied. Particularly in analysis. Before we formally define it, we begin by defining a special type of subset of $\mathscr{P}(S)$ called a "σ-ring".

A subset, $\mathscr{K}$, of $\mathscr{P}(S)$ is called a *σ-ring* if:

1) For any $A \in \mathscr{K}$, $S \backslash A \in \mathscr{K}$
2) Whenever $\{A_i : i \in \mathbb{N}\} \subseteq \mathscr{K}$, then $\cup\{A_i : i \in \mathbb{N}\} \in \mathscr{K}$.

To summarize, a σ-ring is simply a family of sets which is "closed under complements" and "countable unions of its sets". See that $\mathscr{P}(S)$ is itself a σ-ring, while τ is not (since τ, by itself, is not closed under complements). But τ may possibly be a proper subset of some σ-ring.

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[9] This is a more specialized topic. It can be omitted without loss of continuity.

The set $\mathscr{P}(S)$ may contain many σ-rings.

Given a topological space, (S, τ), we will consider all those σ-rings in $\mathscr{P}(S)$, which contain τ. To obtain the *unique smallest σ-ring*, $\mathscr{B}$, in $\mathscr{P}(S)$ that contains τ, we then take the intersection of all σ-rings in $\mathscr{P}(S)$ which contain τ,

$$\mathscr{B} = \cap\{\mathscr{K} \subseteq \mathscr{P}(S) : \mathscr{K} \text{ is a } \sigma\text{-ring}, \tau \subseteq \mathscr{K}\}$$

The reader should verify that this "intersection of all σ-rings in $\mathscr{P}(S)$ which contain τ" is itself a σ-ring containing τ; we emphasize, that this intersection is the *unique* and *smallest* such σ-ring. We have a name for this particular set.

Definition 3.10 Given a topological space (S, τ), we call the smallest σ-ring, $\mathscr{B}$, in $\mathscr{P}(S)$ which contains τ,...

"*the family of Borel sets in S*"

Each member, $A \in \mathscr{B}$, is referred to as a *Borel set*. That is,

$$\{A = \text{"a } \textit{Borel set} \text{ in } S\text{" }\} \Leftrightarrow \{A \in \mathscr{B}\}$$

Every topological space, S, has its own unique family, $\mathscr{B}$, of Borel sets. We identify a Borel set by confirming that it belongs to $\mathscr{B}$. To help us identify Borel sets we list a few properties of $\mathscr{B}$. The reader is left to verify that $\mathscr{B}$ ...

- is closed under complements,
- is closed under countable unions and countable intersections
- contains all G_δ's and all F_σ's.

The definition of a "Borel set" in S makes it difficult to recognize such subsets of S. The following theorem is useful when trying to identify Borel sets.

Theorem 3.11 Let (S, τ) be a topological space. The family, $\mathscr{B}$, of Borel sets is the *unique smallest subfamily of $\mathscr{P}(S)$*, that

(a) contains τ,

(b) is closed under complements

(c) is closed under countable unions.

Furthermore, $\mathscr{B}$ satisfies the following three properties:

1. $\mathscr{B}$ contains all F_σ's of S,
2. $\mathscr{B}$ is closed under countable intersections,
3. $\mathscr{B}$ contains all G_δ's of S.

Proof: Given: (S, τ) is a topological space.

Suppose $\mathscr{B}$ is the family of Borel sets in $\mathscr{P}(S)$. By definition, $\mathscr{B}$ is the intersection of all σ-rings that

(a) contain τ,

(b) that are closed under complements and

(c) closed under countable unions.

So $\mathscr{B}$ is itself the *unique smallest* σ-ring of subsets of S which satisfies these three properties.

Since $\mathscr{B}$ contains τ, it contains all open subsets of S and since it is closed under complements, it contains all closed subsets of S. Since it is closed under countable unions, then it must contain all F_σ's. This establishes property 1.

We now verify that $\mathscr{B}$ is closed under countable intersections: Let $\{A_i : i = 1, 2, 3, \ldots\}$ be a countable family of subsets in $\mathscr{B}$. Then

$$\begin{aligned} \cap\{A_i : i = 1, 2, 3, \ldots\} &= S\backslash S\backslash(\cap\{A_i : i = 1, 2, 3, \ldots\}) \\ &= S\backslash\cup[S\backslash\{A_i : i = 1, 2, 3, \ldots\}] \\ &\in \mathscr{B} \end{aligned}$$

This establishes property 2.

It then follows that, since $\mathscr{B}$ contains all open sets, it follows from property two that it must also contain all G_δ's of S. This establishes property 3.

The above theorem guarantees that every open set, closed set, G_δ and F_σ in a topological space S can be referred to as a Borel set in S. It is sometimes difficult to identify subsets of a topological space (S, τ) which are not Borel sets (with respect to τ). Consider for example, the topological space (S, τ_i) equipped with the indiscrete topology. If A is a non-empty proper subset of S, then A is not a Borel set since $\{\varnothing, S\} = \tau_i$ is the smallest σ-ring which contains τ_i and does not contain the element A. On page 71 of this text we provide another example.

3.7 Free union of topological spaces.

Suppose we are given a family of topological spaces. There is a way to unite them into one single new larger topological space without altering their individual topology. This is referred to a being the "free union" of these topological spaces. We define this concept.

Definition 3.12 Let $\{S_i : i \in I\}$ be a family of topological spaces. For each space, S_i, we associate a space, $S_i^* = \{i\} \times S_i$ in such a way that S_i^* and S_i are identical except for the fact that $\{i\} \times S_i$ has a label i attached to S_i. This is to guarantee that, if $i \neq j$, then S_i^* and S_j^* are entirely different sets and so $S_i^* \cap S_j^* = \varnothing$. This allows us to view the family, $\{S_i^* : i \in I\}$, as being pairwise disjoint, in the sense that no two spaces have elements in common. In most cases, the S_i's have no elements in common and, with this understanding, we can simply ignore the particular notation S_i^* and simply use S_i to represent each set.

We define the *free topological union of* $\{S_i : i \in I\}$,[10] as being the topological space

$$S = \cup\{S_i : i \in I\}$$

in which

"*U is open in S if and only if $U \cap S_i$ is open for each $i \in I$.*"

This topology, thus defined, is referred to as the

disjoint union topology

When we write

$$S = \sum_{i \in I} S_i$$

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[10] Some texts may refer to this set as *direct sum* or *free sum* or *topological direct sum.*

we mean the set "$S = \cup\{S_i : i \in I\}$ equipped with the direct sum topology". If we are speaking of the set $S = A \cup B$ (with only two sets A and B) equipped with the direct sum topology we often simply write

$$S = A + B$$

Example 12. For each real number $r > 0$, we define $g_r(x) = rx$ with domain $(0, \infty)$. For each $r > 0$, let L_r denote the set,

$$L_r = \{(x, y_r) : y_r = g_r(x),\ x > 0\}$$

Let

$$S = \cup\{L_r : r > 0\}$$

See that the set S is the open first quadrant $A = \{(x, y) : x > 0, y > 0\}$ of $\mathbb{R}^2$. Equip each L_r with the subspace topology inherited from $(\mathbb{R}^2, \tau)$. Equip the set S with the direct sum topology, τ_D.

By definition, the subset U of S is open with respect to the direct sum topology if and only if $U \cap L_r$ is open in L_r, for each $r > 0$. If $L_k \subseteq S$, for any $r > 0$, $L_k \cap L_r$ is either $\varnothing$ or equal to L_k both of which are open in L_k. Then each line L_r is open in S.

Since L_r is not open in the A with the standard topology τ, $\tau_D \not\subseteq \tau$. If $T = S \backslash L_k$, $T \cap L_r = L_r$, for each $r \neq k$, so T is open in S. Then L_r is clopen in S, for each r. Then S is the pairwise disjoint union of the family of clopen subsets $\{L_r : r > 0\}$ in S. On the other hand, if $B_\varepsilon(a, b)$ is an open ball in A with the respect to τ, then $B_\varepsilon(a, b) \cap L_r$ is open in L_r, for each r, so $B_\varepsilon(a, b)$ is open in S with respect to τ_D. Then $\tau \subseteq \tau_D$. So τ is a weaker topology then τ_D on A.

3.7 Final remarks on this introductory chapter on topology.

Some readers may wonder, if topological spaces must, by definition, all satisfy the same three topological axioms O1, O2 and O3, what, in essence, distinguishes one topological space from an other? It is important to understand that point-set topology did not emerge from a vacuum. Topology is defined in reference to "sets". Sets by themselves are distinguished from others based on their particular "set-theoretical properties". So we can conclude that what distinguishes topological spaces from each other is how the topological axioms interact with the particular set-theoretical properties they are acting on. Set-theory along with its own family of set-theoretical axioms (ZFC) is the foundation on which point-set topology is built on. This is why it is good to remember that some occasionally encountered difficulties in the study of point-set topology are rooted in a shallow understanding of set theory.[11]

Concepts review.

1. Given a set S, what does a topology τ on S represent? How does one verify whether a family of subsets is a topology?
2. Given an open subset, U, of a topological space, (S, τ), what is the relationship between U and τ?
3. What are the three open set axioms of a topological space (S, τ)?
4. Given a metric space, (S, ρ), describe a topology on S which is induced by ρ.
5. Describe the *usual topology* on $\mathbb{R}$.
6. Given a point, x, in a topological space, (S, τ), what is a *neighborhood* of x?
7. Given two topologies, τ_1 and τ_2, what does it mean to say that τ_1 is weaker than τ_2?
8. Given two topologies, τ_1 and τ_2, what does it mean to say that τ_1 is finer than τ_2?
9. Given a non-empty set, S, describe the discrete topology on S.
10. Given a non-empty set, S, describe the indiscrete topology on S.
11. Suppose F is a closed subset of the topological space (S, τ). What is the relation between F and τ?
12. What are the three closed set axioms, F1, F2, and F3, of a topological space (S, τ).
13. If S is a non-empty set what do we mean by the *cofinite* or *Zariski* topology on S?

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[11] The first few chapters of *Set theory, An introduction to axiomatic reasoning* by R. André could serve as a suitable study companion in such cases.

14. If T is a subset of the topological space (S, τ) what is the subspace topology on T?
15. Suppose B is a subset of the topological space (S, τ) such that $B \notin \tau$. Describe the topology τ_B which extends τ over B?
16. What does it mean to say that a set is metrizable?
17. What is a G_δ of a topological space? What is an F_σ of a topological space?
18. Describe the family of Borel sets in a topological space (S, τ).
19. Provide a few examples of Borel sets in $\mathbb{R}$ equipped with the usual topology. Is $\mathbb{Q}$ a Borel set? Why?

EXERCISES

1. Prove the statement in Theorem 3.5.
2. Consider the open interval $S = (-3, 7)$ in $\mathbb{R}$.
   (a) Construct a topology τ on S which contains five elements.
   (b) Consider the subset $T = (-2, 4] \subset S$. For the topology τ constructed in part (a), what is the subspace topology τ_T on T inherited from S.
   (c) Are the open subsets of T necessarily open subsets of S?
3. Consider $\mathbb{R}$ equipped with the usual topology τ (induced by the Euclidian metric). Let $(\mathbb{Q}, \tau_{\mathbb{Q}})$ be the set of all rational numbers equipped with the subspace topology inherited from $\mathbb{R}$. Consider the subset $T = [-\pi, \pi) \cap \mathbb{Q}$. Determine whether T is open in $\mathbb{Q}$, closed in $\mathbb{Q}$, both open and closed in $\mathbb{Q}$, or none of these.
4. Construct a topology other than the *discrete* or *indiscrete topology* on the set $S = \{\triangle, \Diamond, \Box\}$.
5. Let $\mathscr{F} = \{A \subseteq \mathbb{R} : A \text{ is countable}\} \cup \{\varnothing, \mathbb{R}\}$. Show that $\mathscr{F}$ satisfies the three conditions F1, F2 and F3 described on page 41. Then use this to construct a topology on $\mathbb{R}$. (This is referred to as being the *cocountable topology.*

6. Suppose τ_A and τ_B are two topologies on a set S. Determine whether $\tau_A \cap \tau_B$ is a topology on S.

7. If $\mathbb{R}$ is equipped with the usual topology and $\mathbb{Z}$ represents the set of all integers determine whether $\mathbb{Z}$ is open or closed (or both or neither) in $\mathbb{R}$.

8. Suppose $\mathbb{R}$ is equipped with the usual topology and $T = [1, 4] \cup (6, 10) \subset \mathbb{R}$ where T is equipped with the subspace topology. Determine whether $[1, 4]$ is open in T, closed in T or both open and closed in T. Determine whether $(6, 10)$ is open in T, closed in T or both open and closed in T.

4 / Set closures, interiors and boundaries.

Abstract. *In this section, we introduce the notions of closure and interior of subsets of a topological space. The concept of the boundary of a set is then defined in terms of its interior and closure. Based on their properties, we derive the "closure axioms" and "interior axioms". We then begin viewing closure and interior of sets as being operators on $\mathscr{P}(S)$. From this perspective, we better see how closure and interior operators on $\mathscr{P}(S)$ can be used to topologize a set, providing examples on how this can be done.*

4.1 The closure of a set.

If $T = (2,7]$ is viewed as a subset of the topological space $\mathbb{R}$ (equipped with the usual topology), we easily see that it is not closed since its complement, $\mathbb{R}\backslash T = (-\infty, 2] \cup (7, \infty)$, is not open in $\mathbb{R}$. And yet we feel that it wouldn't take very much for us to "make it closed": We need only add the element, 2, to T to obtain the closed subset $T^* = [2,7]$. Adding the *fewest number* of points possible to a set T to obtain a closed set is what we will refer to as obtaining the *closure of* T. The key words here are "fewest number" of points, and no more. In this case, we would say that the "closure of $T = (2,7]$ is the set $T^* = [2,7]$, the smallest closed subset of $\mathbb{R}$ which contains all the elements of T. With this example in mind, we will now formally define a concept called "closure of a subset".

Definition 4.1 Let S be a topological space and $T \subseteq S$. We define the *closure of* T *in* S, denoted by, $\text{cl}_S T$ (or by $\text{cl}_S(T)$), as

$$\text{cl}_S T = \cap\{F : F \text{ is closed in } S \text{ and } T \subseteq F\}$$

The reader should first be aware of the following verifiable *facts* for any subset, T, of the topological space S.

1) *The closure of* T*,* $\text{cl}_S T$*, is closed in* S: This follows from the fact that arbitrary intersections of closed sets are closed.
2) *The set* $T \subseteq \text{cl}_S T$: This follows from the definition of closure of T.

3) *The set,* $\mathrm{cl}_S T$, *is the smallest closed set which contains* T: Suppose A is a closed set containing T. Then $A \in \{F : F \text{ is closed in } S \text{ and } T \subseteq F\}$. Hence $\mathrm{cl}_S T \subseteq A$.
4) *If* T *is closed, then* $T = \mathrm{cl}_S T$: This is true since T is the smallest closed set containing T.

Let A be a subset of a topological space S.

If x is a point in S such that, for every S-open neighborhood U of x, $U \cap A$ contains some point *other than* x, then we say that...

x is a *cluster point of* A

The set of all cluster points of A is called the *derived set of* A.
This definition provides us with another way of describing a closed set.

> "*The set* B *is closed if and only if it contains all its cluster points.*"

Verification of this fact is left to the reader.

For example, if $B = (1, 3) \cup (3, 5] \cup \{6\}$ in $\mathbb{R}$, the derived set, (that is, the set of all cluster points of B) is $[1, 5]$. The element, 6, is not a cluster point of B since there is an open neighborhood, $(5.5, 7)$ of $\{6\}$ which does not meet other elements of B. The set B is not closed since it doesn't contain the cluster points 1 and 3.

Example 1. Let T be the open interval, $(0, 1)$, viewed as a subset of $\mathbb{R}$ equipped with the usual topology. Then $\mathrm{cl}_{\mathbb{R}} T = [0, 1]$. To prove this we must show that:

1) $[0, 1]$ is closed by showing that $\mathbb{R}\backslash[0, 1]$ is open.
2) $\{0, 1\} \subseteq A$ for any closed set A containing the interval $(0, 1)$.

This is left as an exercise.

Example 2. If $\mathbb{Q}$ is the set of all rational numbers, then $\mathrm{cl}_{\mathbb{R}}(\mathbb{Q}) = \mathbb{R}$.

Proof: To prove this we will show that, if F is a closed subset of $\mathbb{R}$ such that $\mathbb{Q} \subseteq F$, then $F = \mathbb{R}$.

Suppose F is a closed subset of $\mathbb{R}$ such that $\mathbb{Q} \subseteq F$. Then $\mathbb{R}\backslash F$ is open. Suppose $\mathbb{R}\backslash F \neq \varnothing$. Then $\mathbb{R}\backslash F$ is the union of open intervals each of which must contain a rational number. Since $\mathbb{Q} \cap (\mathbb{R}\backslash F) \neq \varnothing$, this contradicts $\mathbb{Q} \subseteq F$. Then $\mathbb{R}\backslash F = \varnothing$. This means that the only closed set containing $\mathbb{Q}$ is $\mathbb{R}$. So $\mathrm{cl}_{\mathbb{R}}(\mathbb{Q}) = \mathbb{R}$.

Example 3. Suppose S is a topological space induced by the metric ρ (that is, the elements of τ are unions of open balls of the form $B_\varepsilon(x) = \{y : \rho(x,y) < \varepsilon\}$). Suppose F is a non-empty subset of S. We define

$$\rho(x,F) = \inf\{\rho(x,u) : u \in F\}$$

Show that $\mathrm{cl}_S F = \{x : \rho(x,F) = 0\}$.

Solution: To do this, we must show

1. $F \subseteq \{x : \rho(x,F) = 0\}$,
2. $S \setminus \{x : \rho(x,F) = 0\}$ is open in S,
3. If $F \subseteq A$ where A is a closed subset of S, then $\{x : \rho(x,F) = 0\} \subseteq A$.

The details are left as an exercise.

We now list and prove a few of the most fundamental closure properties.

Theorem 4.2 Let A and B be two subsets of a topological space (S,τ). Then,

1) $\mathrm{cl}_S(\varnothing) = \varnothing$.

2) If $A \subseteq B$, then $\mathrm{cl}_S(A) \subseteq \mathrm{cl}_S(B)$

3) $\mathrm{cl}_S(A \cup B) = \mathrm{cl}_S(A) \cup \mathrm{cl}_S(B)$ (Closure "distributes" over finite unions.)

4) $\mathrm{cl}_S(\mathrm{cl}_S(A)) = \mathrm{cl}_S(A)$

Proof:

1) Since $\varnothing$ is closed $\mathrm{cl}_S(\varnothing) \subseteq \varnothing$. Since $\varnothing \subseteq \mathrm{cl}_S\varnothing$, then $\mathrm{cl}_S(\varnothing) = \varnothing$.

2) We are given that $A \subseteq B$. If F is closed in S and $B \subseteq F$, then $A \subseteq B \subseteq F$. Then

$$A \subseteq \cap\{F : F \text{ is closed in } S \text{ and } B \subseteq F\} = \mathrm{cl}_S(B)$$

By 1) of the *facts* above, $\mathrm{cl}_S(B)$ is closed in S and so

$$\mathrm{cl}_S(A) = \cap\{F : F \text{ is closed in } S \text{ and } A \subseteq F\} \subseteq \mathrm{cl}_S(B)$$

We have shown that $\mathrm{cl}_S(A) \subseteq \mathrm{cl}_S(B)$.

3) Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, then $\mathrm{cl}_S(A) \subseteq cl_S(A \cup B)$ and $\mathrm{cl}_S(B) \subseteq cl_S(A \cup B)$ (by parts 1) and 2)). So

$$\mathrm{cl}_S(A) \cup cl_S[B) \subseteq \mathrm{cl}_S(A \cup B)$$

Since $A \subset \mathrm{cl}_S(A)$ and $B \subset \mathrm{cl}_S(B)$, $A \cup B \subseteq \mathrm{cl}_S(A) \cup \mathrm{cl}_S(B)$, a closed subset in S. Since $\mathrm{cl}_S(A \cup B)$ is the smallest closed set containing $A \cup B$, then

$$\mathrm{cl}_S(A \cup B) \subseteq \mathrm{cl}_S(A) \cup \mathrm{cl}_S[B)$$

We conclude that $\mathrm{cl}_S(A \cup B) = \mathrm{cl}_S(A) \cup \mathrm{cl}_S(B)$.

4) By part 1) $A \subseteq \mathrm{cl}_S(A) \subseteq \mathrm{cl}_S(\mathrm{cl}_S(A))$. Since $\mathrm{cl}_S(A)$ is closed (see fact 1)), $\mathrm{cl}_S(\mathrm{cl}_S(A)) \subseteq \mathrm{cl}_S(A)$. It then follows that $\mathrm{cl}_S(\mathrm{cl}_S(A)) = \mathrm{cl}_S(A)$.

Example 4. Closure does not distribute over intersections. Suppose $A = (2, 5)$ and $B = (5, 7)$. Then $\mathrm{cl}_{\mathbb{R}}(A \cap B) = \mathrm{cl}_{\mathbb{R}}(\varnothing) = \varnothing$. On the other hand, $\mathrm{cl}_{\mathbb{R}}(A) = [2, 5]$ and $\mathrm{cl}_{\mathbb{R}}(B) = [5, 7]$ hence $\mathrm{cl}_{\mathbb{R}}(A) \cap \mathrm{cl}_{\mathbb{R}}(B) = \{5\}$. This shows that $\mathrm{cl}_S(A) \cap \mathrm{cl}_S(B) \neq \mathrm{cl}_S(A \cap B)$ may sometimes occur.

It is however possible to prove that

$$\mathrm{cl}_S(A \cap B) \subseteq \mathrm{cl}_S(A) \cap \mathrm{cl}_S(B)$$

Proving this is left as an exercise.[1]

Remark on closures of arbitrary unions. We have seen in Theorem 4.2 that $\mathrm{cl}_S(A \cup B) = \mathrm{cl}_S A \cup \mathrm{cl}_S B$, so the "closure distributes over finite unions".

This does not hold true for arbitrary unions. If it is always true that, for a family of subsets, $\{A_i : i \in I\}$ of S,

$$\cup\{\mathrm{cl}_S A_i : i \in I\} \subseteq \mathrm{cl}_S[\cup\{A_i : i \in I\}]$$

it may occur that $\mathrm{cl}_S[\cup\{A_i : i \in I\}]$ is not contained in $\mathrm{cl}_S[\cup\{A_i : i \in I\}]$.

Consider, for example, the sets of the form

$$A_i = \left(\frac{1}{i}, 3\right] \text{ where } \mathrm{cl}_{\mathbb{R}} A_i = \left[\frac{1}{i}, 3\right]$$

---

[1] However, note that, if A is open in S and $W \subseteq S$, $\mathrm{cl}_S W \cap A \subseteq \mathrm{cl}_S(A \cap W)$. To see this, note that, since A is open in S, $\mathrm{cl}_S W \cap A = [\mathrm{cl}_S W \setminus A \cup \mathrm{cl}_S(W \cap A)] \cap A = [\mathrm{cl}_S W \setminus A \cap A] \cup [\mathrm{cl}_S(W \cap A)] \cap A] = \varnothing \cup [\mathrm{cl}_S(W \cap A)] \cap A] \subseteq \mathrm{cl}_S(W \cap A)$.

for $i = 1, 2, 3, \ldots$ Verify that $\cup\{\mathrm{cl}_{\mathbb{R}}(A_i) : i = 1, 2, 3, \ldots\} = (0, 3]$ (left as an exercise).

Since $\mathrm{cl}_{\mathbb{R}}[\cup\{A_i : i = 1, 2, 3, \ldots\}] = [0, 3]$ (left as an exercise), then

$$\cup\{\mathrm{cl}_{\mathbb{R}}(A_i) : i = 1, 2, 3, \ldots\} \neq \mathrm{cl}_{\mathbb{R}}[\cup\{A_i : i = 1, 2, 3, \ldots\}]$$

Note that, under certain specific conditions, it may occur that the "closure distributes over unions". For example, a bit later in text, we show (in Lemma 6.17) that, for any collection, $\{A_i : i \in I\}$, of sets said to be "locally finite", $\cup\{\mathrm{cl}_S A_i : i \in I\} = \mathrm{cl}_S[\cup\{A_i : i \in I\}]$.

4.2 Closure viewed as an operator on $\mathscr{P}(S)$.

The closure of a set can be viewed as an action

$$\mathrm{cl}_S : \mathscr{P}(S) \to \mathscr{P}(S)$$

performed on a set. It takes an arbitrary set, A, and associates to it another set, $\mathrm{cl}_S(A)$, obtained by adding sufficiently many points (but no more) so as to produce a "closed set". It can then be viewed as a function. With this in mind, we define the *Kuratowski closure operator.*

Kuratowski closure operator. Suppose S is a non-empty set and $K : \mathscr{P}(S) \to \mathscr{P}(S)$ is a function which satisfies the four conditions:

K1. $K(\varnothing) = \varnothing$ and $A \subseteq K(A)$
K2. If $A \subseteq B$, then $K(A) \subseteq K(B)$
K3. $K(A \cup B) = K(A) \cup K(B)$
K4. $K(K(A)) = K(A)$

A function, $K : \mathscr{P}(S) \to \mathscr{P}(S)$, satisfying these four properties is referred to as a *Kuratowski closure operator* where K1 to K4 are the *Kuratowski closure operator axioms.*

Topologizing a set S by using a closure operator. The reader should notice that, in the above definition, the set S is not described as being a "topological space" since no topology is defined on it. It is just a set. The following theorem shows that, if we are given a Kuratowski operator, $K : \mathscr{P}(S) \to \mathscr{P}(S)$, on $\mathscr{P}(S)$, then we can use K to generate a topology, τ_K, on S such that

$$\mathrm{cl}_S A = K(A)$$

for all $A \in \mathscr{P}(S)$.

Theorem 4.3 Let S be a set and suppose $K : \mathscr{P}(S) \to \mathscr{P}(S)$ satisfies the four *Kuratowski closure operator axioms.* Define $\mathscr{F} \subseteq \mathscr{P}(S)$ as

$$\mathscr{F} = \{A \subseteq S : K(A) = A\}$$

a) Then $\mathscr{F}$, is the set of all closed subsets of some topology, τ_K, on S, i.e.,

$$\tau_K = \{S \backslash A : \text{ where } A \in \mathscr{F}\}$$

b) Furthermore, in (S, τ_K), $\text{cl}_S(A) = K(A)$, for any $A \subseteq S$.

Proof: Given: The operator $K : \mathscr{P}(S) \to \mathscr{P}(S)$. Also, $\mathscr{F} = \{A \subseteq S : K(A) = A\}$.

a) To prove the statement(a) it will suffice to show that $\mathscr{F}$ satisfies the three "closed sets" conditions F1, F2 and F3 on page 41. If so, then we can define τ_K by invoking the statement of the proposition on page 41.

- Note that $\{\varnothing, S\} \subseteq \mathscr{F}$. To see this note that, by K1, $S \subseteq K(S) \subseteq S \Rightarrow K(S) = S$. Both $\varnothing$ and S belong to $\mathscr{F}$. The set $\mathscr{F}$ satisfies condition F1.
- Suppose $\mathscr{U} = \{F_i : i \in I\}$ is an arbitrarily large family of sets in $\mathscr{F}$. By K1, $\cap\mathscr{U} \subseteq K(\cap\mathscr{U})$. Also, by K2, $K(\cap\mathscr{U}) \subseteq K(F_i) = F_i$ for each $i \in I$. So $K(\cap\mathscr{U}) \subseteq \cap\{F_i : i \in I\} = \cap\mathscr{U}$. Hence $K(\cap\mathscr{U}) \subseteq \cap\mathscr{U}$. So $K(\cap\mathscr{U}) = \cap\mathscr{U}$. We then have $K(\cap\mathscr{U}) \in \mathscr{F}$. The set $\mathscr{F}$ is then closed under arbitrary intersections. The set $\mathscr{F}$ satisfies condition F2.
- We now show that $\mathscr{F}$ is closed under finite unions. We must show that, if A and $B \in \mathscr{F}$, then $K(A \cup B) = A \cup B$. Consider $A, B \in \mathscr{F}$. By K2,

$$A \cup B \quad \subseteq \quad K(A \cup B)$$

$$\begin{aligned} A \cup B \subseteq K(A) \cup K(B) \quad &\Rightarrow \quad K(A \cup B) \subseteq K(K(A) \cup K(B)) \quad \text{(K2.)} \\ &\Rightarrow \quad K(A \cup B) \subseteq K(K(A)) \cup K(K(B)) \quad \text{(K3.)} \\ &\Rightarrow \quad K(A \cup B) \subseteq A \cup B \quad \text{(K4.)} \end{aligned}$$

We conclude that $K(A \cup B) = K(A) \cup K(B)$. The set $\mathscr{F}$ satisfies condition F3.

So $\mathscr{F}$ is the set of all closed sets in S. This means the topology τ_K on S is

$$\tau_K = \{S \backslash A : \text{ where } A \subseteq \mathscr{F}\}$$

Hence the set, $\mathscr{F} = \{A \subseteq S : K(A) = A\}$, represents all closed subsets of S (with respect to τ_K).

b) We now prove the second statement, $\mathrm{cl}_S(A) = K(A)$.
Let $A \subseteq (S, \tau_K)$. Then $\mathrm{cl}_S(A) \in \mathscr{F}$. So $K(\mathrm{cl}_S(A)) = \mathrm{cl}_S(A)$. We claim that, from this we can obtain $\mathrm{cl}_S(A) = K(A)$.
Proof of claim: Let $A \subseteq S$.

$$\begin{aligned}
K(K(A)) &= K(A) \quad \text{\scriptsize (By K4.)} \\
&\Rightarrow K(A) \in \mathscr{F} \\
&\Rightarrow S \backslash K(A) \in \tau_K \\
&\Rightarrow K(A) \text{ is closed with respect to } \tau_K \\
&\Rightarrow \mathrm{cl}_S(A) \subseteq K(A) \quad \text{\scriptsize (By K2, } {\scriptstyle A \subseteq K(A)}\text{\scriptsize .)} \\
A \subseteq \mathrm{cl}_S(A) &\Rightarrow K(A) \subseteq K(\mathrm{cl}_S(A)) \quad \text{\scriptsize (By K2.)} \\
&\Rightarrow K(A) \subseteq \mathrm{cl}_S(A) \quad \text{\scriptsize (Since } {\scriptstyle K(\mathrm{cl}_S(A)) = \mathrm{cl}_S(A)}\text{\scriptsize)}
\end{aligned}$$

We conclude that $\mathrm{cl}_S(A) = K(A)$, as claimed.

We have shown that any Kuratowski closure operator $K : \mathscr{P}(S) \to \mathscr{P}(S)$ can be used to construct a topology, τ_K, on S in such a way that, for any$A \subseteq S$, $K(A) = \mathrm{cl}_S(A)$. We illustrate this in the following example.

Example 5. We consider the set $\mathbb{R}^2$, a set with uncountably many elements. We define a function $K : \mathscr{P}(\mathbb{R}^2) \to \mathscr{P}(\mathbb{R}^2)$ as follows:

$$\begin{aligned}
K(A) &= A \text{ if } A \text{ is countable} \\
K(A) &= \mathbb{R}^2 \text{ if } A \text{ is uncountable} \\
K(\varnothing) &= \varnothing
\end{aligned}$$

Show that K is a Kuratowski operator. Then find the topology on $\mathbb{R}^2$ induced by the operator K.

Solution. Proving that K satisfies the properties K1 to K4 is routine and so is left as an exercise. Then, thus defined, K is a Kuratowski closure operator. By Theorem 4.3, the set

$$\mathscr{F} = \{A \subseteq \mathbb{R}^2 : K(A) = A\} = \{A \subseteq \mathbb{R}^2 : A \text{ is countable}\} \cup \{\varnothing, \mathbb{R}^2\}$$

represents the set of all closed subsets of the topological space $(\mathbb{R}^2, \tau_K)$. We deduce that

$$\tau_K = \{B \subseteq \mathbb{R}^2 : \mathbb{R}^2 \backslash B \text{ is countable}\} \cup \{\varnothing, \mathbb{R}^2\}$$

We will refer to τ_K in this example as the *cocountable topology* on $\mathbb{R}^2$.

4.3 The interior of a set.

Given a non-empty set A the "closure of A" has been defined as being the intersection of all closed subsets of S which contain A. We now wish to consider the union of all open subsets of S which are entirely contained in A.

Definition 4.4 Let A be a non-empty subset of the topological space (S, τ). We say that a point x is an *interior point* of A if there exist an open subset, U, of S such that $x \in U \subseteq A$. We define the *interior of A*, denoted $\text{int}_S A$ (or as $\text{int}_S(A)$) as follows:

$$\text{int}_S A = \{x \in S : x \in U \subseteq A \text{ for some open } U \text{ in } S.\ \}$$

If A contains no interior points, then we will say that the interior, $\text{int}_S A$, of A is empty.

Clearly $\text{int}_S A \subseteq A$. For example, if $A = (2, 4] \cup \{5\}$, $\text{int}_{\mathbb{R}} A = (2, 3)$. The element 5 does not belong to $\text{int}_{\mathbb{R}} A$ since there is no open interval U in $\mathbb{R}$ such that $5 \in U \subseteq A$. The fact that $\{5\}$ is open in A is irrelevant.

The reader is left to verify that $\text{int}_S A$ can equivalently be described as being...

"*the largest open subset of S which is entirely contained in A*"

The following theorem shows a relationship between the interior and closure of a set. It also proposes a method to determine the interior of a set, A, by considering the closure of its complement, $S \backslash A$.

Theorem 4.5 Let (S, τ) be a topological space and A be a subset of S. Then,

$$S\backslash\text{int}_S(A) = \text{cl}_S(S\backslash A)$$

Proof: Given: (S, τ) is a topological space and A is a subset of S.

Since $\text{int}_S(A) \subseteq A$, then $S\backslash A \subseteq S\backslash\text{int}_S(A)$. But $S\backslash\text{int}_S(A)$ is closed in S, hence

$$\text{cl}_S(S\backslash A) \subseteq S\backslash\text{int}_S(A)$$

Also,

$$\begin{aligned} S\backslash A \subseteq \text{cl}_S(S\backslash A) &\Rightarrow S\backslash(\text{cl}_S(S\backslash A)) \subseteq A \\ &\Rightarrow S\backslash(\text{cl}_S(S\backslash A)) \subseteq \text{int}_S(A) \\ &\Rightarrow S\backslash\text{int}_S(A) \subseteq (\text{cl}_S(S\backslash A)) \end{aligned}$$

We thus obtain $S\backslash\text{int}_S(A) = \text{cl}_S(S\backslash A)$

Using this theorem, we let the reader verify that the following three statements are equivalent:

(a) $\text{int}_S(A) = S\backslash\text{cl}_S(S\backslash A)$,
(b) $\text{int}_S(S\backslash A) = S\backslash\text{cl}_S(A)$,
(c) $\text{cl}_S(A) = S\backslash(\text{int}_S(S\backslash A))$

Just as for the closure of a set we have four basic similar properties for the interior of sets.

Theorem 4.6 Let (S, τ) be a topological space and A and B be subsets of S.

(a) The set, $\text{int}_S(A)$, is open in S. Also, $\text{int}_S(A)$ is the largest open subset of S which is entirely contained in A.

(b) If $B \subseteq A$, then $\text{int}_S(B) \subseteq \text{int}_S(A)$.

(c) The set $\text{int}_S(A \cap B) = \text{int}_S(A) \cap \text{int}_S(B)$. ($\text{Int}_S$ "distributes" over finite intersections.)

(d) The set $\text{int}_S(\text{int}_S(A)) = \text{int}_S(A)$.

Proof: The proofs of statements(a), (b) and (d) are left as an exercise.

Proof of $\text{int}_S(A \cap B) = \text{int}_S(A) \cap \text{int}_S(B)$:

$$\begin{array}{rcl} S\backslash\text{int}_S(A \cap B) & = & \text{cl}_S(S\backslash(A \cap B)) \\ & = & \text{cl}_S[(S\backslash A) \cup (S\backslash B)] \\ & = & \text{cl}_S(S\backslash A) \cup \text{cl}_S(S\backslash B) \\ & = & [S\backslash\text{int}_S(A)] \cup [S\backslash\text{int}_S(B)] \\ & = & S\backslash[\text{int}_S(A) \cap \text{int}_S(B)] \\ & \Rightarrow & \\ \text{int}_S(A \cap B) & = & \text{int}_S(A) \cap \text{int}_S(B) \end{array}$$

Example 6. Given that $\mathbb{R}$ is equipped with the usual topology, what is $\text{int}_{\mathbb{R}}(\mathbb{Q})$?

Solution: We consider the subset, $\mathbb{Q}$, of all rationals in $\mathbb{R}$. By Theorem 4.6 part (a), $\text{int}_{\mathbb{R}}(\mathbb{Q}) \subseteq \mathbb{Q}$. If $\text{int}_{\mathbb{R}}(\mathbb{Q})$ is non-empty, it should be a union of non-empty open intervals. But every open interval in $\mathbb{R}$ contains an irrational; then $\text{int}_{\mathbb{R}}(\mathbb{Q}) = \varnothing$.

Example 7. If $\mathbb{R}$ is equipped with the usual topology, then $\text{int}_{\mathbb{R}}([0,1]) = (0,1)$.

Example 8. The set $\text{int}_{\mathbb{R}}(A \cup B)$ need *not* be equal to $\text{int}_{\mathbb{R}}(A) \cup \text{int}_{\mathbb{R}}(B)$: If $\mathbb{R}$ is equipped with the usual topology, then

$$\text{int}_{\mathbb{R}}\,(\,[0,1] \cup [1,2]\,) = (0,2)$$

while

$$\text{int}_{\mathbb{R}}[0,1] \ \cup \text{int}_{\mathbb{R}}[1,2] = (0,1) \cup (1,2)$$

4.4 The interior viewed as an operator $\text{int}_S : \mathscr{P}(S) \to \mathscr{P}(S)$.

Just as for closures of sets we can view "int_S" as a function, $I : \mathscr{P}(S) \to \mathscr{P}(S)$.
Let S be a non-empty set and $I : \mathscr{P}(S) \to \mathscr{P}(S)$ be a function satisfying the properties:

I1. $I(S) = S$
I2. $I(A) \subseteq A$, for all $A \subset S$,

I3. $I(I(A)) = I(A)$, for all $A \subset S$,

I4. $I(A \cap B) = I(A) \cap I(B)$, for all $A, B \in \mathscr{P}(S)$

The function $I : \mathscr{P}(S) \to \mathscr{P}(S)$ satisfying the listed properties is called an *interior operator*. We refer to I1 to I4 as being the *interior operator axioms.*

Topologizing a set S by using an interior operator. Again, just as for the closure operator, the definition of the function, $I : \mathscr{P}(S) \to \mathscr{P}(S)$, doesn't refer to any topology on S. But we will show that the function, I, can be used to define a topology on S by choosing appropriate sets in its range. We desire a topology, τ, such that A is open in S if and only if $A = \text{int}_S A = I(A)$.

Theorem 4.7 Let S be a set and suppose $I : \mathscr{P}(S) \to \mathscr{P}(S)$ satisfies the four *interior operator axioms.* Define $\mathscr{U} \subseteq \mathscr{P}(S)$ as

$$\mathscr{U} = \{A \subseteq S : I(A) = A\}$$

Then,

(a) The set $\mathscr{U}$ forms a topology on S.

(b) Furthermore, if S is equipped with topology $\mathscr{U}$, $\text{int}_S(A) = I(A)$, for any $A \subseteq S$.

Proof: Let S be a set and $I : \mathscr{P}(S) \to \mathscr{P}(S)$ be a function satisfying the four *interior operator axioms* I1 to I4 listed above. Let $\mathscr{U} = \{A \in \mathscr{P}(S) : I(A) = A\}$.

(a) We are required to prove that $\mathscr{U}$ forms a topology on S.

We see that:

- By I1, $I(S) = S$, so $S \in \mathscr{U}$. By I2, $I(\varnothing) \subseteq \varnothing$. Since $\varnothing \subseteq I(\varnothing)$, then $I(\varnothing) = \varnothing$ and so $\varnothing \in \mathscr{U}$
- Suppose A and B belong to $\mathscr{U}$. By property I4, $I(A \cap B) = I(A) \cap I(B) = A \cap B$. So $\mathscr{U}$ is closed under finite intersections.
- To show that $\mathscr{U}$ is closed under arbitrary unions we first verify that
  $(A \subseteq B) \Rightarrow (I(A) \subseteq I(B)) \quad (*)$

$$\begin{aligned} A \subseteq B &\Rightarrow A = B \cap A \\ &\Rightarrow I(A) = I(B \cap A) = I(B) \cap I(A) \\ &\Rightarrow I(A) \subseteq I(B) \end{aligned}$$

Let $\{A_i\}_{i\in M}$ be a collection of sets in $\mathscr{U}$. It suffices to show $I(\cup\{A_i\}_{i\in M}) = \cup\{A_i\}_{i\in M}$.

$$\begin{array}{rcll} I(A_i) \subseteq \cup\{I(A_i)\}_{i\in M} & \Rightarrow & I(I(A_i)) \subseteq I(\cup\{I(A_i)_{i\in M}) & \text{(By *.)} \\ & \Rightarrow & I(A_i) \subseteq I(\cup\{I(A_i)\}_{i\in M}) & \\ & \Rightarrow & \cup\{I(A_i)\}_{i\in M} \subseteq I(\cup\{I(A_i)\}_{i\in M}) & \end{array}$$

By I2, $I(\cup\{I(A_i)\}_{i\in M}) \subseteq \cup\{I(A_i)\}_{i\in M}$. Then $I(\cup\{I(A_i)\}_{i\in M}) = \cup\{I(A_i)\}_{i\in M}$ so $\cup\{I(A_i)\}_{i\in M} \in \mathscr{U}$.

Then set $\mathscr{U}$ satisfies the three open set axioms O1, O2, and O3. So $\mathscr{U}$ is a topology on S, as required.

We will denote the topology $\mathscr{U}$ on S induced by the operator I, by τ_I.

(b) We are now required to show that $I(A) = \text{int}_S(A)$ with respect to τ_I.
Suppose $A \subseteq S$.

By I3, $I(I(A)) = I(A)$, so $I(A) \in \mathscr{U} = \tau_I$; so $I(A)$ is open. Since, $\text{int}_S(A)$ is the largest open subset of S contained in A, and $I(A) \subseteq A$ (by I2),

$$I(A) \subseteq \text{int}_S(A)$$

Also, see that, since $\text{int}_S(A) \in \tau_I = \mathscr{U}$ and $\text{int}_S(A) \subseteq A$,

$$\begin{array}{rcll} \text{int}_S(A) & = & I(\text{int}_S(A)) & \\ & \subseteq & I(A) & \text{(By * above } A \subseteq B \Rightarrow I(A) \subseteq I(B)) \end{array}$$

then $\text{int}_S(A) \subseteq I(A)$.
We conclude that $I(A) = \text{int}_S(A)$.

We provide a few examples.

Example 9. Let $I : \mathscr{P}(\mathbb{R}) \to \mathscr{P}(\mathbb{R})$ be a function defined as

$$\begin{array}{rcll} I(\mathbb{R}) & = & \mathbb{R} & \\ I(A) & = & A\backslash\mathbb{Q} & \text{otherwise.} \end{array}$$

(a) Show that $I : \mathscr{P}(\mathbb{R}) \to \mathscr{P}(\mathbb{R})$, thus defined, is an interior operator on $\mathscr{P}(\mathbb{R})$.

(b) Use the interior operator described in part (a) to define a topology, τ_I, on $\mathbb{R}$.

(c) For the topology, τ_I, on $\mathbb{R}$ shown in part (b), describe the open subsets, the closed subsets of $\mathbb{R}$, the closure of sets and the interior of sets.

(d) A *Borel sets* example: If $\mathscr{F}$ represents the set of all closed subsets with respect to the topology, τ_I, on $\mathbb{R}$, show that

$$\mathscr{B} = \tau_I \cup \mathscr{F}$$

is the smallest σ-ring containing τ_I and so is a family of Borel sets.

Solution.

(a) We show I is an interior operator.

1. By definition, $I(\mathbb{R}) = \mathbb{R}$ so I1 is satisfied.
2. Also, if $A \neq \mathbb{R}$, $I(A) = A \backslash \mathbb{Q} \subseteq A$. So I2 is satisfied.
3. If $A \neq \mathbb{R}$

$$\begin{aligned} I(I(A)) &= I(A)\backslash\mathbb{Q} \\ &= (A\backslash\mathbb{Q})\backslash\mathbb{Q} \\ &= A\backslash\mathbb{Q} \\ &= I(A) \end{aligned}$$

So $I(I(A)) = A$. So I3 is satisfied.

4. If neither A nor B is $\mathbb{R}$,

$$\begin{aligned} I(A \cap B) &= (A \cap B)\backslash\mathbb{Q} \\ &= (A\backslash\mathbb{Q}) \cap (B\backslash\mathbb{Q}) \\ &= I(A) \cap I(B) \end{aligned}$$

So $I(A \cap B) = I(A) \cap I(B)$. Then I4 is satisfied.

This means that $I : \mathscr{P}(\mathbb{R}) \to \mathscr{P}(\mathbb{R})$ is an interior operator.

(b) We now use the interior operator described in part (a) to topologize $\mathbb{R}$.

Since $I : \mathscr{P}(\mathbb{R}) \to \mathscr{P}(\mathbb{R})$ has been shown to be an interior operator, then

$$\begin{aligned} \tau_I &= \{A : I(A) = A\} \\ &= \{A : A\backslash\mathbb{Q} = A\} \cup \{\mathbb{R}\} \\ &= \{A : A \text{ does not contain any points of } \mathbb{Q}\} \cup \{\mathbb{R}\} \end{aligned}$$

is a topology on $\mathbb{R}$.

(c) For the topology, τ_I, on $\mathbb{R}$ we now describe the open subsets, the closed subsets of $\mathbb{R}$, the closure of sets and the interior of sets.

Open sets in $\mathbb{R}$. Open sets in $\mathbb{R}$, are $\mathbb{R}$ itself and all sets which do not contain any rationals, including $\varnothing$.

For example, if $\mathbb{J}$ is the set of irrationals and $r \in \mathbb{J}$, then, since $\{r\}$ contains no rationals, $\{r\}$ is an open singleton set. Also, if $q \in \mathbb{Q}$, $\mathbb{R}$ is the only open set containing q. So $\{q\}$ is *not* an open singleton set.

Closed sets in $\mathbb{R}$. Suppose B is not $\mathbb{R}$. We claim that B is closed in $\mathbb{R}$ with respect to τ_I if and only if $\mathbb{Q} \subseteq B$:

$$\begin{aligned}
\mathbb{Q} \subseteq B \quad &\Leftrightarrow \quad \mathbb{R}\backslash B = (\mathbb{R}\backslash B)\backslash \mathbb{Q} \\
&\Leftrightarrow \quad \mathbb{R}\backslash B = I(\mathbb{R}\backslash B) \\
&\Leftrightarrow \quad \mathbb{R}\backslash B \text{ is open } {\scriptstyle (\text{with respect to } \tau_I)} \\
&\Leftrightarrow \quad B \text{ is closed } {\scriptstyle (\text{with respect to } \tau_I)}
\end{aligned}$$

For example, if $r \in \mathbb{J}$, since $\mathbb{Q} \not\subseteq \{r\}$, the singleton set, $\{r\}$, is *not* a closed set.

Closure of a set. Then taking the closure of a subset A of $\mathbb{R}$ comes down to adding all of $\mathbb{Q}$ to A. That is, if $A \neq \mathbb{R}$, $\mathrm{cl}_{\mathbb{R}} A = A \cup \mathbb{Q}$. For example, if $q \in \mathbb{Q}$, since $\mathrm{cl}_{\mathbb{R}}\{q\} = \mathbb{Q}$, $\{q\}$ is *not* closed.

Interior of a set in $\mathbb{R}$. $\mathrm{Int}_{\mathbb{R}} A = A \cap \mathbb{J}$. Finding the interior of A comes down to removing any trace of $\mathbb{Q}$ in A.

(d) We are required to show: The set, $\mathscr{B} = \tau_I \cup \mathscr{F}$, is the smallest σ-ring containing τ_I and so is a family of Borel sets of $\mathbb{R}$.

Given: $\mathscr{F}$ is the set of all closed subsets in $\mathbb{R}$ with respect to τ_I. We claim that $\mathscr{B}$ is a σ-ring.

Closure of $\mathscr{B}$ *under countable unions.* If $U \in \tau_I$ and $V \in \mathscr{F}$, then $U \cup V \in \mathscr{F}$ (since an open subset union a closed subset containing $\mathbb{Q}$ contains $\mathbb{Q}$. So $U \cup V$ is a closed subset.)

The set $\mathscr{F}$ is closed under countable unions (since closed subsets are those subsets of $\mathbb{R}$ which contain all of $\mathbb{Q}$, arbitrary unions of elements of $\mathscr{F}$ are closed with respect to τ_I). This actually show that $\mathscr{F}$ contains all its F_σ's. Since all F_σ's are closed, then all G_δ's are open and so belong to τ_I.

Trivially, τ_I is closed under arbitrary unions and so is closed under countable unions.

Closure of $\mathscr{B}$ under "complements". Let $U \in \mathscr{B}$. If $U \in \tau_I$, then $\mathbb{R}\backslash U \in \mathscr{F}$. If $U \in \mathscr{F}$, then $\mathbb{R}\backslash U \in \tau_I$.

So $\mathscr{B}$ is a σ-ring, as claimed. Since it must contains τ_I and all complements it is the smallest σ-ring containing τ_I. By definition, it is a family of Borel sets of $\mathbb{R}$ with respect to τ_I.[2]

4.5 The boundary of the subset of a space.

We have seen that, for any subset, A, of a topological space (S, τ),

$$\text{int}_S(A) \subseteq A \subseteq \text{cl}_S(A)$$

Often, $\text{cl}_S(A) \setminus \text{int}_S(A) \neq \varnothing$. We will now briefly discuss those sets whose points belong to $\text{cl}_S(A)\backslash\text{int}_S(A)$.

Definition 4.8 Let A be a subset of a topological space (S, τ). We define the *boundary of* A, denoted as $\text{bd}_S(A)$, as

$$\text{bd}_S(A) = \text{cl}_S(A)\backslash\text{int}_S(A)^3$$

The expressions, $\text{Fr}_S(A)$, $\partial_S(A)$, $\text{Bd}_S(A)$ are also commonly used to represent the "boundary of A". It is easily verified that

$$\text{bd}_S(A) = \text{cl}_S(A) \cap \text{cl}_S(S\backslash\text{int}_S A)$$

Since the finite intersection of closed sets is closed, we see that the boundary, $\text{bd}_S(A)$ of a set A, is always closed. Furthermore, for any set A in S, both A and $S \setminus A$ share the same boundary (like adjacent neighbours who share the same fence). It is always the case that $\text{int}_S(A) \cap \text{bd}_S(A) = \varnothing$ and that $\text{int}_S(A)$, $\text{bd}_S(A)$ and $\text{int}_S(\text{cl}_S(S\setminus A))$ are pairwise disjoint sets whose union is S. The reader is left to verify this.

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[2] Note that, if T is the closed interval [1, 3] in $\mathbb{R}$, then $T \notin \tau_I \cup \mathscr{F}$ (since $[1, 3]$ contains some elements of $\mathbb{Q}$, but does not contain all of $\mathbb{Q}$) and so is not an element of the unique family, $\mathscr{B}$, of all Borel sets, with respect to τ_I. We can then refer to it as a "*non-Borel set*".

[3] The word "Frontier" is also sometimes used instead of "boundary". When the term "Frontier of A" is used, it is denoted by $\text{Fr}_S A$.

Example 10. If $\mathbb{Q}$ is viewed as a subset of $\mathbb{R}$ equipped with the usual topology, then

$$\begin{aligned} \mathrm{bd}_{\mathbb{R}}(\mathbb{Q}) &= \mathrm{cl}_{\mathbb{R}}(\mathbb{Q})\backslash\ \mathrm{int}_{\mathbb{R}}(\mathbb{Q}) \\ &= \mathbb{R}\backslash\varnothing \\ &= \mathbb{R} \end{aligned}$$

Example 11. If $B = [0,1]$ is a closed interval viewed as a subset of $\mathbb{R}$ equipped with the usual topology, then $\mathrm{bd}_{\mathbb{R}}(B) = \{0,1\}$. It is left to the reader to verify this.

Example 12. Let $B = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ be a subset of $\mathbb{R}^2$ equipped with the topology induced by the Euclidean metric. Then

$$\mathrm{bd}_{\mathbb{R}^2}(B) = \mathrm{cl}_{\mathbb{R}^2}(B)\backslash\mathrm{int}_{\mathbb{R}^2}(B) = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$$

The reader is left to verify the details.

Example 13. Suppose the set of natural numbers $\mathbb{N}$ is equipped with the cofinite topology.[4] Let E denote the set of all even natural numbers. Then

$$\begin{aligned} \mathrm{bd}_{\mathbb{N}}(E) &= \mathrm{cl}_{\mathbb{N}}(E)\backslash\mathrm{int}_{\mathbb{N}}(E) \\ &= \mathbb{N}\backslash\varnothing \\ &= \mathbb{N} \end{aligned}$$

Example 14. Suppose $B = (0,1) \cup (1,2]$ viewed as a subset of $\mathbb{R}$ equipped with the usual topology. We compute the boundary to be,

$$\begin{aligned} \mathrm{bd}_{\mathbb{R}}(B) &= \mathrm{cl}_{\mathbb{R}}(B)\backslash\mathrm{int}_{\mathbb{R}}(B) \\ &= [0,2] \setminus (0,1) \cup (1,2) \\ &= \{0,2\} \end{aligned}$$

It is interesting to note that

$$\begin{aligned} \mathrm{int}_{\mathbb{R}}(\mathrm{cl}_{\mathbb{R}}(B)) &= \mathrm{int}_{\mathbb{R}}([0,2]) \\ &= (0,2) \neq B \end{aligned}$$

---

[4] The open subsets are those whose complement is finite.

4.6 On dense subsets of a topological space.

Suppose B and A are subsets of (S, τ) such that $A \subseteq B$ and $B \backslash A$ contains only boundary points of A. When two sets relate to each other in this way we say that "A is dense in B". We define this formally.

Definition 4.9 Suppose A and B are subsets in a topological space (S, τ). If

$$A \subseteq B \quad \text{and} \quad B \subseteq \text{cl}_S(A)$$

then we will say that

"*A is a dense subset of B*"

In the case where $B = S$, then A is dense in S if and only if $\text{cl}_S A = S$.
If $A \subseteq S$ is such that $\text{int}_S \text{cl}_S A = \varnothing$, then we say that A is *nowhere dense in* S.

Example 15. Suppose

$A = \{(x, y) : x^2 + y^2 < 1\}$ and $B = A \cup \{(0, 1), (1, 0), (0, -1), (-1, 0)\}$

Since

$$A \subseteq B \text{ and } B \subset \ \text{cl}_{\mathbb{R}^2} A = \{(x, y) : x^2 + y^2 \leq 1\}$$

then A is dense in B.

Example 16. *A nowhere dense subset.* Let $A = \{6\}$ be a subset of $\mathbb{R}$. Then $\text{cl}_{\mathbb{R}} A = \{6\}$ and $\text{int}_S \text{cl}_S A = \varnothing$. So A is nowhere dense in the space $\mathbb{R}$.

Another example: The set of all integers, $\mathbb{Z}$, is nowhere dense in $\mathbb{R}$, since $\text{int}_{\mathbb{R}} \text{cl}_{\mathbb{R}} \mathbb{Z} = \varnothing$.

Example 17. Verify that the set $C = \{(x, y) : x^2 + y^2 = 1\}$ is nowhere dense in $\mathbb{R}^2$.

On countable dense subsets of a space.

The property in the following definition is one which refers to an upper bound for the cardinality of a dense subset of a topological space. Hence, in a way it expresses a restriction on its size.

Definition 4.10 We will say that a topological space (S, τ) is *separable* if and only if S contains a countable dense subset.

For example, the topological space, $(\mathbb{R}, \tau)$, equipped with the usual topology is *a separable space* since, $\text{cl}_{\mathbb{R}}\mathbb{Q} = \mathbb{R}$, and the subset of all rational numbers is a countable subset of $\mathbb{R}$. Notice how the border of $\mathbb{Q}$ is much larger than $\mathbb{Q}$ itself.

We will subdivide the class of all topological spaces into two subclasses: Those that are separable and those that are not.

4.7 Topic: Regular open sets and regular closed sets.

Suppose A is a non-empty open subset of a topological space (S, τ). We know that $A \subseteq \text{cl}_S A$ is always true. Also, it is always true that, if A is open, $A = \text{int}_S A \subseteq \text{int}_S \text{cl}_S A$ (by part (b) of Theorem 4.6). So it is always true that

$$A \in \tau \quad \Rightarrow \quad A \subseteq \text{int}_S \text{cl}_S A \qquad (\dagger)$$

One may wonder whether, when A is open, must we have equality between A and $\text{int}_{\mathbb{R}}\text{cl}_{\mathbb{R}}A$. The following simple example will help answer this question.

Let $A = (1, 3) \cup (3, 7)$, an open subset of $\mathbb{R}$. Then, by applying the above reasoning we have,

$$A = (1, 3) \cup (3, 7) \subseteq \text{int}_{\mathbb{R}}\text{cl}_{\mathbb{R}}A = \text{int}_{\mathbb{R}}[1, 7] = (1, 7)$$

So in the case where $A = (1, 7)$ we do have equality; but if $A = (1, 3) \cup (3, 7)$ we have an open subset, A, such that $A \neq \text{int}_{\mathbb{R}}\text{cl}_{\mathbb{R}}A$. Then the best we can then do is affirm that, when A is open, A is a subset of $\text{int}_S\text{cl}_S A$. We have a special name for those open subsets where equality holds.

Definition 4.11 An open subset, A, of a topological space (S, τ) is called a *regular open subset of* S if and only if $A = \text{int}_S \text{cl}_S A$. In this book, we denote the set of all regular open subsets of S by $\mathscr{Ro}(S)$

We know that for an open set A, it is always true that, $A \subseteq \text{int}_S\text{cl}_S A$. Then

$$\begin{aligned} S\backslash(\text{int}_S\text{cl}_S A) \subseteq S\backslash A &\Rightarrow \text{cl}_S[\,S\backslash(\text{cl}_S A)\,] \subseteq S\backslash A \\ &\Rightarrow \text{cl}_S\text{int}_S(S\backslash A) \subseteq S\backslash A \end{aligned}$$

So, it is always true that,

$$F \text{ closed } \Rightarrow \text{cl}_S\text{int}_S(F) \subseteq F$$

Then, if A is regular open, $A = \text{int}_S\text{cl}_S A$, so for the complement, $F = S\backslash A$, we can say that

$$F = \text{cl}_S\text{int}_S(F)$$

So $F = \text{cl}_S\text{int}_S(F)$ if and only if the complement of F is regular open.

We have a name for the complements of regular open subsets.

Definition 4.12 A closed subset, F, of a topological space (S, τ) is called a *regular closed subset of* S if and only if

$$F = \text{cl}_S\,\text{int}_S F$$

Hence the regular closed subsets of S are precisely the complements of the regular open subsets. We denote the set of all regular closed subsets of S by $\mathscr{R}c(S)$.

Example 18. Let S be a topological space and $\mathscr{R}o(S)$ be the set of all regular open sets in S.

(a) Verify that $\mathscr{R}o(S)$ is closed under finite intersections but is not closed under finite unions.

(b) We know that, if $A = \text{int}_S\text{cl}_S A$, then A belongs to $\mathscr{R}o(S)$. Suppose A is some non-empty subset of S which does not belong to $\mathscr{R}o(S)$. Verify that $\text{int}_S\text{cl}_S A$ belongs to $\mathscr{R}o(S)$.

(c) If $\mathscr{R}c(S)$ denotes the set of all regular closed sets in S, show that $\mathscr{R}c(S)$ is closed under finite unions.

Solution: Given: $\mathscr{Ro}(S)$ is the set of all regular open sets in S.

(a) Suppose A and B are open subsets. Then $A \cap B$ is open. Suppose $A = \text{int}_S\text{cl}_S A$ and $B = \text{int}_S\text{cl}_S B$. Then

$$\begin{aligned} \text{int}_S\text{cl}_S(A \cap B) &\subseteq \text{int}_S(\text{cl}_S A \cap \text{cl}_S B) \quad \text{By Theorem 4.6} \\ &= \text{int}_S\text{cl}_S A \cap \text{int}_S\text{cl}_S B \\ &= A \cap B \end{aligned}$$

Since $A \cap B \subseteq \text{int}_S\text{cl}_S(A \cap B)$ and $\text{int}_S\text{cl}_S(A \cap B) \subseteq A \cap B$, then $A \cap B$ is regular open.

So the family of all regular open sets is closed under finite intersections.

On the other hand, $(1, 5)$ and $(5, 9)$ are both regular open but $(0, 5) \cup (5, 9)$ is not. So the family of all regular open sets is not closed under finite unions.

(b) We are given $A \subseteq S$. We are required to show that $\text{int}_S\text{cl}_S A$ is regular open. See that

$$\begin{aligned} \text{int}_S\text{cl}_S(\text{int}_S\text{cl}_S A) &\subseteq \text{int}_S\text{cl}_S(\text{cl}_S A) \quad (\text{Since } \text{int}_S\text{cl}_S A \subseteq \text{cl}_S A) \\ &= \text{int}_S\text{cl}_S A \end{aligned}$$

Since $\text{int}_S\text{cl}_S A \subseteq \text{int}_S\text{cl}_S(\text{int}_S\text{cl}_S A)$, then $\text{int}_S\text{cl}_S A$ is regular open.

(c) This part is left an exercise.

Example 19. Let S be a topological space. Show that, if D is a dense subset of S and V is an open subset in S then $\text{cl}_S V = \text{cl}_S(V \cap D)$.

Solution: Since $V \cap D \subseteq V$ then $\text{cl}_S(V \cap D) \subseteq \text{cl}_S V$. Suppose, on the other hand, that, $x \in \text{cl}_S V \setminus \text{cl}_S(V \cap D)$. Then, for any S-open neighborhood U of x, $U \cap V \neq \varnothing$. The open subset U, can also be chosen so that $U \cap \text{cl}_S(V \cap D) = \varnothing$. Then $U \cap (V \cap D) = (U \cap V) \cap D = \varnothing$. Since D is dense in S and $U \cap V \cap D$ cannot be empty. We have a contradiction. So

$$\text{cl}_S V = \text{cl}_S(V \cap D)$$

Example 20. Let S be a topological space. Suppose U, F are two subsets of S such that $U \subset F$. Verify that $\text{cl}_F U = F \cap \text{cl}_S U$. In the particular case that F is closed in S verify that $\text{cl}_F U = \text{cl}_S U$.

Solution: Since $\text{cl}_F U \subseteq F$ and $\text{cl}_F U \subseteq \text{cl}_S U$ then $\text{cl}_F U \subseteq F \cap \text{cl}_S U$. If $x \in (F \cap \text{cl}_S U) \setminus \text{cl}_F U$, there exists an S-open neighborhood V of $x \in F$ such that $V \cap \text{cl}_S U \neq \varnothing$ and $V \cap \text{cl}_F U = \varnothing$. Then $V \cap U \neq \varnothing$ and

$V \cap U = \varnothing$, a contradiction. So $\mathrm{cl}_F U = F \cap \mathrm{cl}_S U$. If F is closed in S then, $F \cap \mathrm{cl}_S U = \mathrm{cl}_S U$ (since $\mathrm{cl}_S U \subseteq \mathrm{cl}_S F = F$). We have verified the following result:

$$[\,U \subseteq F_{\text{closed}} \subseteq S\,] \Rightarrow [\,\mathrm{cl}_F U = \mathrm{cl}_S U\,]$$

Concepts review.

1. Given a topological space (S, τ) and $T \subseteq S$, define the closure, $\mathrm{cl}_S(T)$, in S, with respect to the topology τ.
2. Does cl_S "distribute" over finite unions? How about finite intersections?
3. List the four Kuratowski closure operator axioms, K1 to K4.
4. If $K : \mathscr{P}(S) \to \mathscr{P}(S)$ satisfies the four Kuratowski closure operator axioms describe a topology on S which can be constructed from K.
5. Define the cocountable topology on $\mathbb{R}^2$. Describe the Kuratowski operator used to develop this topology.
6. Given a topological space (S, τ) and $A \subseteq S$, define "interior point" of A with respect to τ. Define the interior, $\mathrm{int}_S(A)$, of A with respect to τ.
7. Does int_S distribute over finite unions? How about finite intersections?
8. State the four interior operator axioms I1 to I4 for $I : \mathscr{P}(S) \to \mathscr{P}(S)$.
9. Given a set S, describe a topology that can be constructed on S by using the operator $I : \mathscr{P}(S) \to \mathscr{P}(S)$.
10. Describe the topology induced on $\mathbb{R}$ by the operator $I(A) = A \backslash \mathbb{Q}$.
11. Define the boundary of a set with respect to a topology τ on S.
12. Define what we mean when we say that "A is dense in B".
13. Define what we mean when we say that "A is nowhere dense in B".
14. Define a regular open subset and a regular closed subset of a space.
15. Show that, for any subset U of S, $\mathrm{int}_S \mathrm{cl}_S U$ is regular open in S.

EXERCISES

1. Let $T = (0, 1)$, viewed as a subset of $\mathbb{R}$ equipped with the usual topology. Show that $\mathrm{cl}_{\mathbb{R}}(T) = [0, 1]$. This is Example 1 on page 58.
2. Suppose S is a topological space induced by the metric ρ (that is, the elements of τ are unions of open balls of the form $B_\varepsilon(x) = \{y : \rho(x, y) < \varepsilon\}$). Suppose F is a non-empty subset of S. We define $\rho(x, F) = \inf\{\rho(x, u) : u \in F\}$. Show that $\mathrm{cl}_S(F) = \{x : \rho(x, F) = 0\}$. This is Example 3 on page 59.
3. Consider $\mathbb{R}$ equipped with the usual topology τ (induced by the Euclidian metric). Let $(\mathbb{Q}, \tau_{\mathbb{Q}})$ be the set of all rational numbers equipped with the subspace topology inherited from $\mathbb{R}$. Consider the subset $T = [-\pi, \pi) \cap \mathbb{Q}$. Determine whether T is open in $\mathbb{Q}$, closed in $\mathbb{Q}$, both open and closed in $\mathbb{Q}$, or none of these.
4. In Example 4 on page 60 it is shown that $\mathrm{cl}_S(A) \cap \mathrm{cl}_S(B) \neq \mathrm{cl}_S(A \cap B)$ sometimes occurs. Show that $\mathrm{cl}_S(A \cap B) \subseteq \mathrm{cl}_S(A) \cap \mathrm{cl}_S(B)$ is always true.
5. Let (S, τ) be a topological space and A and B be subsets of S. Show that:
   (a) $\mathrm{int}_S(A)$ is open in S. Also, $\mathrm{int}_S(A)$ is the largest open subset of S which is entirely contained in A.
   (b) If $B \subseteq A$, then $\mathrm{int}_S(B) \subseteq \mathrm{int}_S(A)$.
   (c) $\mathrm{int}_S(A \cap B) = \mathrm{int}_S(A) \cap \mathrm{int}_S(B)$.
   (d) $\mathrm{int}_S(\mathrm{int}_S(A)) = \mathrm{int}_S(A)$.

   (This is Theorem 4.6. Part (c) is already proven.)
6. Let S be a set and $I : \mathscr{P}(S) \to \mathscr{P}(S)$ be an interior operator satisfying the four conditions I1 to I4 listed on page 66. Show that, if $\tau = \{A \in \mathscr{P}(S) : I(A) = A\}$, (S, τ) is a topological space such that $I(A) = \mathrm{int}_S(A)$ for all $A \in \mathscr{P}(S)$.
7. If $B = [0, 1]$ is a closed interval viewed as a subset of $\mathbb{R}$ equipped with the usual topology show that $\mathrm{bd}_{\mathbb{R}}(B) = \{0, 1\}$.
8. Let $B = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1, x > 0\} \cup \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1, x \leq 0\}$ be a subset of $\mathbb{R}^2$ equipped with the topology induced by the Euclidean metric. Show that

$$\mathrm{bd}_{\mathbb{R}^2}(B) = \mathrm{cl}_{\mathbb{R}^2}(B) \backslash \mathrm{int}_{\mathbb{R}^2}(B) = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$$

9. Show that A is both open and closed in the topological space (S, τ) if and only if $\text{bd}_S(A)$ is empty.

10. Let A and B be two subsets of the topological space (S, τ). Show that, if $\text{bd}_S(A) \cap \text{bd}_S(B) = \varnothing$, then $\text{int}_S(A \cup B) = \text{int}_S(A) \cup \text{int}_S(B)$.

11. Let $B = [0, 7) \cup \{9\}$ where B is equipped with the subspace topology inherited from $\mathbb{R}$ itself equipped with the usual topology. If $A = \{0\} \cup \{3\} \cup (5, 7)$, find (i) $\text{cl}_B(A)$, (ii) $\text{bd}_B(A)$, iii) $\text{int}_B(A)$.

5 / Bases of topological spaces.

Abstract. *In this section we define a neighborhood system of $x \in S$ with respect to a given topology on S. Given a topology, τ, on S, we define a "base for the topology τ". We deduce a set-theoretic property called the "base property" possessed by any base. Those subsets of $\mathscr{P}(S)$ which satisfy the described property will be shown to be a base for some topology on S. We introduce the notion of a "subbase for the topology τ" by describing its properties. We then show how to topologize a set both from a collection of sets which possesses the "base property" and also from an arbitrary collection of subsets. In the last part of this section we introduce a miscellany of topological spaces in which open bases and neighborhood bases play a particularly important role. The introduction of these spaces this early in the text explains why it is slightly longer than one might expect. Familiarity with theses spaces will allow us to more easily refer to these in various chapters later in the text.*

5.1 Neighborhoods of points.

It is not always easy to confirm that a given subset, $\mathscr{S}$, of $\mathscr{P}(S)$ is a well-defined topology, τ, on a set S. Ultimately, we would prefer a topology on S to be described in a way that provides some intuitive idea about what its open subsets are like. In this section we describe a subfamily, $\mathscr{B}$, of open sets from which every open set in τ is constructed. With this objective in mind, we begin by introducing the concept of "*neighborhoods*" of a point in (S, τ).

Definition 5.1 Let (S, τ) be a topological space and $x \in A \subseteq S$. We will say that A is a *neighborhood of x with respect to τ* if $x \in \text{int}_S(A)$. For a given $x \in S$, the set

$$\mathscr{U}_x = \{A \in \mathscr{P}(S) : A \text{ is a neighborhood of } x\}$$

is called a *neighborhood system of x with respect to τ*.

A subfamily, $\mathscr{B}_x$, of a neighborhood system, $\mathscr{U}_x$, such that, for any set, $A \in \mathscr{U}_x$, there exists $B_x \in \mathscr{B}_x$ such that $x \in B_x \subseteq A$, is called a

neighborhood base at x

The elements of a neighborhood base, $\mathscr{B}_x$ at x, are called *basic neighborhoods*. The basic neighborhoods need not be open in the space (S, τ); but if every element of the neighborhood base, $\mathscr{B}_x$, belongs to τ, then we specify this by declaring $\mathscr{B}_x$ to be an

open neighborhood base at x

.

Note that, since the definition of a neighborhood refers to the "interior" of sets, a *neighborhood system* is always expressed with respect to some topology, τ, defined on the set S. Distinguish the two concepts defined in the above statement. We see that a neighborhood base, $\mathscr{B}_x$, at x is a subset of a neighborhood system at x. The set, $\mathscr{B}_x$, must be such that, any open neighborhood of x, contains an element of $\mathscr{B}_x$. While there can be only one neighborhood system at x we will see that there can be more than one neighborhood base at this point.

For example, $B = [-1, 5) \cup [6, \infty)$ is a neighborhood of 0 with respect to the usual topology of $\mathbb{R}$. The set

$$\mathscr{U}_3 = \{U \in \mathscr{P}(\mathbb{R}) : 3 \in \text{int}_{\mathbb{R}}(U)\}$$

is a neighborhood system of 3. If A belongs to $\mathscr{U}_3$, the element "3" must belong to the interior of A in the sense that 3 cannot be sitting on A's boundary. For example, $[0, 4] \in \mathscr{U}_3$, but $[1, 3] \notin \mathscr{U}_3$. Notice that, for a given $x \in S$, its neighborhood system, $\mathscr{U}_x$, with respect to τ is, by definition, unique.

Based on the definition, we can make the following comments about neighborhoods of a point in a topological space, (S, τ).

- The empty set, $\varnothing$, is not the neighborhood of any point, so $\varnothing \notin \mathscr{U}_x$ for any $x \in S$.
- In general, a neighborhood, U, of a point x is not necessarily open, but it must contain x in its interior, $\text{int}_S(U)$.
- If $x \in S$, then $\mathscr{U}_x$ is not empty since S is a neighborhood of x. If τ is the *indiscrete topology*, $\{\varnothing, S\}$, then S is the only neighborhood of every point $x \in S$.
- In the metric space $(\mathbb{R}, \tau)$ where τ is the usual topology, it is easily verified that both

$$\begin{aligned} \mathscr{B}_1(0) &= \{(-\varepsilon, \varepsilon) : \varepsilon > 0\} \\ \mathscr{B}_2(0) &= \{[-\varepsilon, \varepsilon] : \varepsilon > 0\} \end{aligned}$$

satisfy the definition of a neighborhood base at the point 0. The set $\mathscr{B}_1(0)$ is specified to be an "open neighborhood base at the point 0".

The "neighborhood of a point" concept allows us to come up with another characterization of open sets provided we have predefined "interior of a set". We propose: "[A is open in S] $\Leftrightarrow$ [$x \in A \Rightarrow \exists$ neighborhood, U_x, where $x \in U_x \subseteq A$]". This will work since this would imply that

$$A = \cup_{x \in A}\{\text{int}_S(U_x) : U_x \in \mathscr{U}_x \text{ and } U_x \subseteq A\}$$

Or, we could simply say that "A is open in S provided A contains an x-neighborhood, U_x, for each point x in A". This definition of "open set" (in terms of neighborhoods) is equivalent to its formal definition of "open set".

For a non-empty set $M \subseteq S$, the "neighborhood of a point" concept allows us to describe $\text{cl}_S M$ as follows:

$$\text{cl}_S M = \{x \in S : \text{every neighborhood of } x \text{ intersects } M\}$$

Clearly, the set on the right side must be a subset of $\text{cl}_S M$. Also, if $y \in M$, y clearly must belong to the set on the right-hand side. If $y \in \text{cl}_S M \setminus M$ then every neighborhood of y must intersect M. So $\text{cl}_S M$ is a subset of the set on the right-hand side.

5.2 A base for a topology.

In our study of normed vector spaces (as well as of metric spaces) we have seen that, introducing the notion of "open ball, $B_\varepsilon(x)$, center x with radius ϵ", led to a convenient way of constructing a subfamily, $\mathscr{B}_x = \{B_\varepsilon(x) : \varepsilon > 0\}$, of open neighborhoods of a point x. Every open subset of S can then simply be described as being the union of open neighborhoods of the form $B_\varepsilon(x)$ contained in the set. The collection of open sets,

$$\mathscr{B} = \cup\{\mathscr{B}_x : x \in S\}$$

is sufficient to generate all open subsets of S.

Definition 5.2 Let (S, τ) be a topological space. Suppose $\mathscr{B}$ is a subset of τ satisfying the property:

> "For any $A \in \tau$, A is the union of elements of a subset, $\mathscr{C}$, of $\mathscr{B}$."

We call the subset, $\mathscr{B}$, a *base for open sets* or an *open base for the topology* τ (often abbreviated by simply saying *a base for* S). The word *basis* is sometimes used instead of "base".[1]

The elements of a base are referred to as *basic open sets.*[2]

If $\mathscr{F}$ is a family of closed subsets of S satisfying the property:

> "For any closed subset B in S, B is the intersection of elements of a subset, $\mathscr{C}$, of $\mathscr{F}$."

we call the subset, $\mathscr{F}$, a *base for closed sets* of S.

A base for open sets is generally not unique. Given the topological space $(\mathbb{R}, \tau)$, where τ is the usual topology, both τ and $\mathscr{B} = \{(a, b) : a < b\}$ are bases for $\mathbb{R}$.

It is easily verified that ...

> *If $\mathscr{F}$ is a base for closed subsets of S, then the family, $\mathscr{B}$, of all complements of the elements of $\mathscr{F}$ will form a base for open sets.*

How does one go about constructing a useful base for a topology? A good way to start is to determine some properties possessed by a known useful open base. The following theorem characterizes a base for a topology on a set S.

Theorem 5.3 Let (S, τ) be a topological space and $\mathscr{B} \subset \tau$. Then the following are equivalent:

1. The family $\mathscr{B}$ is a base for τ.
2. For any $x \in S$, $\mathscr{B}$ contains an open neighborhood base, $\mathscr{B}_x$, of sets at x.
3. If $U \in \tau$ and $x \in U$, there exists $B \in \mathscr{B}$ such that $x \in B \subseteq U$.

---

[1] Both words "base" and "basis" are commonly used; thus the reader can assume these have the same meaning.

[2] Note that, if $A = \varnothing \in \tau$, then A is the union of all elements from $\mathscr{C} = \varnothing = \{\ \} \subseteq \mathscr{B}$. So $\mathscr{B}$ also generates the empty set.

Proof: We are given that (S, τ) is a topological space and $\mathscr{B} \subset \tau$.

(1 ⇒ 2) Suppose the set $\mathscr{B}$ is a base for a topology τ on S and $x \in S$. Let

$$\mathscr{B}_x = \{B \in \mathscr{B} : x \in B\}$$

Suppose A is an open subset of S which contains x. It suffices to show that $\mathscr{B}_x$ contains an open neighborhood U of x such that $x \in U \subseteq A$. By hypothesis and definition 5.2, $A = \cup\{B : B \in \mathscr{B}\}$. Then, there exists some $B_x \in \mathscr{B}$ such that $x \in B_x \subseteq A$. By definition, $B_x \in \mathscr{B}_x$. So $\mathscr{B}_x$ is an open neighborhood base of x contained in $\mathscr{B}$.

(2 ⇒ 3) This follows immediately from the definition of "neighborhood base".

(3) ⇒ (1) Suppose $A \in \tau$. If A is empty, then A is the union of all open sets in $\varnothing = \{ \} \in \mathscr{B}$. Suppose $x \in A$. By hypothesis, there exists $B \in \mathscr{B}$ such that $x \in B \subseteq A$. Then A is the union of sets from $\mathscr{B}$. So $\mathscr{B}$ is a base for open subsets of S.

5.3 The "base property".

We have seen how an arbitrary set can be topologized by two different techniques. One by using a *Kuratowski closure operator*, the other by using an *interior operator*. These operators must satisfy certain axioms. If they do, then we can define a particular topology corresponding to the operator. In this section, we will propose two other methods for topologizing a set. We will first discuss another important characterization of a base for a topology on a set S.

Theorem 5.4 Let S be a non-empty set and $\mathscr{B}$ be a non-empty subset of $\mathscr{P}(S)$. The set $\mathscr{B}$ is a *base* for a topology τ on S if and only if $S = \cup\{B : B \in \mathscr{B}\}$ and, if $x \in A \cap B$ for some $A, B \in \mathscr{B}$, then there exists $C \in \mathscr{B}$ such that $x \in C \subseteq A \cap B$.

Proof: We are given that S is a non-empty set and $\mathscr{B} \subseteq \mathscr{P}(S)$.

(⇒) Suppose the set $\mathscr{B}$ is a base for a topology, τ, on S. Since $S \in \tau$, and $\mathscr{B}$ is a base for S, then $S = \cup\{B : B \in \mathscr{C}\}$ for some subset, $\mathscr{C}$, of $\mathscr{B}$. Since $\cup\{B : B \in \mathscr{B}\} \subseteq S$, then $S = \cup\{B : B \in \mathscr{B}\}$.

Suppose $x \in A \cap B$ for some $A, B \in \mathscr{B}$. Then $A, B \in \tau$ and so $A \cap B \in \tau$. Since $A \cap B$ is open and (by Theorem 5.3) $\mathscr{B}$ contains an open neighborhood base, $\mathscr{B}_x$, there exists $B_x \in \mathscr{B}_x \subseteq \mathscr{B}$ such that $x \in B_x \subseteq A \cap B$. So B_x plays the role of C is the statement, as required.

($\Leftarrow$) We are given the non-empty set, S, and the subset, $\mathscr{B}$ of $\mathscr{P}(S)$ which satisfies the two properties: $S = \cup\{B : B \in \mathscr{B}\}$ and if $x \in A \cap B$ for some $A, B \in \mathscr{B}$, then there exists $C \in \mathscr{B}$ such that $x \in C \subseteq A \cap B$.

Let

$$\mathscr{T} = \{A \subseteq S : A = \cup\{C : C \in \mathscr{C}\} \text{ for some subset } \mathscr{C} \text{ of } \mathscr{B}\}$$

We are required to show that $\mathscr{T}$ is a topology on S and that $\mathscr{B}$ is base for $\mathscr{T}$. It suffices to show that $\mathscr{T}$ is a topology on S for, if so, by definition, $\mathscr{B}$ is a base for $\mathscr{T}$.

O1 The set S belongs to $\mathscr{T}$: Since $S = \cup\{B : B \in \mathscr{B}\}$, then $S \in \mathscr{T}$. We claim the set $\varnothing$ belongs to $\mathscr{T}$: The union of all elements in $\varnothing \subset \mathscr{B}$ is $\varnothing$. So $\varnothing \in \mathscr{T}$.

O2 *Claim*: The set $\mathscr{T}$ is closed under unions. Let $\{A_\alpha : \alpha \in \Gamma\} \subseteq \mathscr{T}$. For $\alpha \in \Gamma$, $A_\alpha = \cup\{B : B \in \mathscr{B}_\alpha \subseteq \mathscr{B}\}$. Then $\cup_{\alpha\in\Gamma}\{A_\alpha\} = \cup_{\alpha\in\Gamma}\{\cup_{B\in\mathscr{B}_\alpha}\{B\}\}$, a union of elements in $\mathscr{B}$. So $\cup_{\alpha\in\Gamma}\{A_\alpha\} \in \mathscr{T}$.

O3 *Claim*: The set $\mathscr{T}$ is closed under finite intersections. Let $A, C \in \mathscr{T}$. It suffices to show that $A \cap C \in \mathscr{T}$. Let $x \in A \cap C$. See that

$$A = \cup\{B : B \in \mathscr{B}_A \subseteq \mathscr{B}\} \text{ and } C = \cup\{B : B \in \mathscr{B}_C \subseteq \mathscr{B}\}$$

There exist $B_A \in \mathscr{B}_A$ and $B_C \in \mathscr{B}_C$ such that $x \in B_A \cap B_C$. By hypothesis, there exists $B_x \in \mathscr{B}$ such that $x \in B_x \subseteq B_A \cap B_C \subseteq A \cap C$. Then for every $x \in A \cap C$ there exists $B_x \in \mathscr{B}$ such that $x \in B_x \subseteq A \cap B$. Then $A \cap B = \cup_{x\in A\cap C}\{B_x\}$; so $A \cap C \in \mathscr{T}$, as required.

So $\mathscr{T}$ is a topology on S. By definition of $\mathscr{T}$, every element of $\mathscr{T}$ is a union of elements of $\mathscr{B}$ so, by definition of "open base", $\mathscr{B}$ is base of $\mathscr{T}$.

The following theorem may also be useful as a tool to topologize a set by creating a base for closed sets for some topology.

Theorem 5.5 Let S be a non-empty set and $\mathscr{F}$ be a non-empty subset of $\mathscr{P}(S)$. If $\cap\{F : F \in \mathscr{F}\} = \varnothing$ and, whenever $A, B \in \mathscr{F}$, $A \cup B$ is the intersection of elements in $\mathscr{F}$ then the set $\mathscr{F}$ is a *base* for the closed sets in S for *some* topology on S.

Proof: We are given that S is a non-empty set and $\mathscr{F} \subseteq \mathscr{P}(S)$.

We are given that the subset, $\mathscr{F}$ of $\mathscr{P}(S)$ satisfies the two properties: $\cap\{F : F \in \mathscr{F}\} = \varnothing$ and whenever $A, B \in \mathscr{F}$, $A \cup B$ is the intersection of elements in $\mathscr{F}$. Let $\mathscr{B} = \{S\backslash F : F \in \mathscr{F}\}$.

Since $\cap\{F : F \in \mathscr{F}\} = \varnothing$ then $\cup\{S\backslash F : F \in \mathscr{F}\} = S$ (for if not, we easily obtain a contradiction of the hypothesis). So $\cup\mathscr{B} = S$.

Suppose $A, B \in \mathscr{F}$. Then, by hypothesis, there exists $\mathscr{C} \subseteq \mathscr{F}$ such that $A \cup B = \cap\mathscr{C}$. Then

$$\begin{array}{rcl} S\backslash A \cap S\backslash B & = & S\backslash[A \cup B] \\ & = & S\backslash[\cap\backslash C] \\ & = & \cup\{S\backslash F : F \in \mathscr{C}\} \end{array}$$

Then there exists $F_1 \in \mathscr{C}$ such that $S\backslash F_1 \subseteq S\backslash A \cap S\backslash B$.
By Theorem 5.4, the family of subsets, $\{S\backslash F : F \in \mathscr{F}\}$, forms a base for open subsets for some topology, τ, on S. So $\mathscr{F}$ forms a base for closed subsets for the topology, τ, on S.

In this text, we will call the special property which is satisfied by $\mathscr{B} \subseteq \mathscr{P}(S)$ in the statement of Theorem 5.4, the "base property". We formally define this below.

Definition 5.6 Let S be a non-empty set and let $\mathscr{B}$ be a subset of $\mathscr{P}(S)$.

Base property : The set $\mathscr{B}$ satisfies the *base property*, if $S = \cup\{B : B \in \mathscr{B}\}$ and, if $x \in A \cap B$ for some $A, B \in \mathscr{B}$, then there exists $C \in \mathscr{B}$ such that $x \in C \subseteq A \cap B$.[3]

The Theorem 5.4 guarantees that, given any non-empty set S, if a subset, $\mathscr{B}$, of $\mathscr{P}(S)$ satisfies the *base property*, then $\mathscr{B}$ forms a base for some topology, τ, on S. The family, τ, is made precisely of the arbitrary unions of the elements of subsets of $\mathscr{B}$. In this case, we say that $\mathscr{B}$ *generates the topology* τ. The above statement is a powerful tool for topologizing sets.

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[3] *Note*: The set $\mathscr{F}$ which appears in Theorem 5.5 will, more specifically, be referred to as the *base property for closed sets.*

Throughout our study of general topology it will be very useful to have a variety of examples of topological spaces at hand. The reader is encouraged to take note of these, or at least bookmark them, for future reference in this book. For this purpose, most are given a particular name so that they can be more easily be referred to in the index. The following three well-known examples illustrate in detail how the above result is used to topologize certain subsets of $\mathbb{R}^2$.

Example 1. Let (S, τ) be some topological space. Recall that a G_δ-set is a subset of a space, (S, τ) which is a countable intersection of elements of τ. Let

$$\mathscr{G} = \{G \in \mathscr{P}(S) : G \text{ is a } G_\delta\text{-set}\} \subseteq \mathscr{P}(S)$$

(the set of all G_δ-sets in a topological space, (S, τ)). Note that an element of $\mathscr{G}$ need not necessarily belong to τ. Show that $\mathscr{G}$ is an open base for some topology, $\tau_{\mathscr{G}}$, on S which is stronger than the topology τ.

Solution: Since every element of τ is an element of $\mathscr{G}$, then $S = \cup\{G : G \in \mathscr{G}\}$. Suppose $A, B \in \mathscr{G}$ and $x \in A \cap B$. Then there exist open subsets, $\{U_i : i \in \mathbb{N}\}$ and $\{V_i : i \in \mathbb{N}\}$, such that $A = \cap\{U_i : i \in \mathbb{N}\}$ and $B = \cap\{V_i : i \in \mathbb{N}\}$. Then

$$\begin{array}{rcl} x & \in & (\cap\{U_i : i \in \mathbb{N}\}) \ \cap \ (\cap\{V_i : i \in \mathbb{N}\}) \\ & = & \cap\{U_i \cap V_i : i \in \mathbb{N}\} \quad \text{(A } G_\delta.) \\ & \subseteq & A \cap B \end{array}$$

Since $\cap\{U_i \cap V_i : i \in \mathbb{N}\}$ is a G_δ, then $\mathscr{G}$ satisfies the base property and so generates a topology, $\tau_{\mathscr{G}}$.

We will refer to $\tau_{\mathscr{G}}$ as the

“*G_δ-topology generated by τ*”

Since every open base element of τ is a G_δ, $\tau \subseteq \tau_{\mathscr{G}}$.

Example 2. The *Moore plane.* Also called, *Niemytzki's topology.*

Let $S = \{(x, y) \in \mathbb{R}^2 : y \geq 0\}$. The set $B_\varepsilon(a, b)$ represents the usual open ball of center, (a, b), and radius ε. Let

$$\mathscr{A} = \{B_\varepsilon(x_0, y_0) : x_0 \in \mathbb{R}, y_0 > 0, \text{ and } \varepsilon < y_0\}$$

For each $x \in \mathbb{R}$ and $\varepsilon > 0$, let

$$D_\varepsilon(x, 0) = \{(x, 0)\} \cup B_\varepsilon(x, \varepsilon)$$

That is, $D_\varepsilon(x,0)$ is an open ball, $B_\varepsilon(x,\varepsilon)$, of radius ε tangent to the horizontal axis at $(x,0)$ with the point $(x,0)$ attached to it. Let

$$\mathscr{D} = \{D_\varepsilon(x,0) : x \in \mathbb{R}, \varepsilon > 0\}$$

Let $\mathscr{B} = \mathscr{A} \cup \mathscr{D}$. Show that $\mathscr{B}$ is the base for some topology on S.

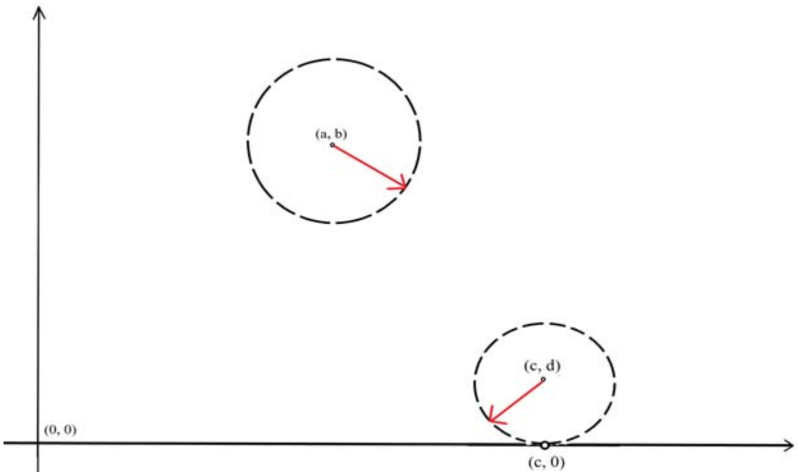

Figure 2: Diagram: Moore plane

Solution: If we show that $\mathscr{B}$ satisfies the "base property", then $\mathscr{B}$ is a base which generates a topology, τ, on the half-plane S. It is first easily seen that $S = \cup\{B : B \in \mathscr{B}\}$.

We consider case 1: Suppose $A, B \in \mathscr{B}$ and $(x,y) \in A \cap B$ where $y > 0$. Then the situation is analogous to what occurs in $\mathbb{R}^2$ with the usual topology, and so

$$(x,y) \in B_\varepsilon(x,y) \subseteq A \cap B$$

for some $\varepsilon > 0$.

We consider case 2: Suppose $A, B \in \mathscr{B}$ and $(x,0) \in A \cap B$. Then $A = \{(x,0)\} \cup B_{\varepsilon_1}(x,\varepsilon_1)$ and $B = \{(x,0)\} \cup B_{\varepsilon_2}(x,\varepsilon_2)$. Let $\varepsilon = \frac{\min\{\varepsilon_1,\varepsilon_2\}}{2}$ and $C = \{(x,0)\} \cup B_\varepsilon(x,\varepsilon)$. Then

$$(x,0) \in C \subseteq A \cap B$$

By invoking Theorem 5.4, we conclude that $\mathscr{B}$ is a base for a topology, τ_M, on S.

This well-known topological space, (S,τ_M), is referred to as the *Moore plane*. Some authors also refer to this topology, τ_M, on S, as the

Niemytzki's tangent disc topology .

The topology τ_M is strictly stronger than τ, the usual topology. To see this, let τ represent the subspace topology on the set S, inherited from $\mathbb{R}^2$ equipped with the usual topology. We verify that τ_M is strictly stronger than τ. Let $B(0,0)$ denote a basic open neighborhood of the point $(0,0)$ in τ_M. Then $B(0,0) \cap \{(0,0)\} = \{(0,0)\}$. The set $B_\varepsilon(0,0) \cap S$ is a half-open disc which contains the interval, $(-\varepsilon, \varepsilon)$, and so cannot be contained in $B(0,0)$. Then $B(0,0) \notin \tau$ so $T_M \not\subseteq \tau$. On the other hand, given $(u,0) \in B_\varepsilon(0,0) \cap S$ one can construct a neighborhood base element, $B(u,0)$, of radius small enough so that it is contained in $B_\varepsilon(0,0) \cap S$. So $\tau \subseteq \tau_M$.

So the Moore plane topology is strictly stronger than the usual topology on S.

Example 3. *The radial plane.* Consider the set $\mathbb{R}^2$. We will equip $\mathbb{R}^2$ with what is called the *radial plane topology.* If $(x,y) \in \mathbb{R}^2$, we define a "radial ball centered at (x,y)", $B_{(x,y)}$, in $\mathbb{R}^2$, as follows:

> *The singleton set, $\{(x,y)\}$, union the set of all open line segments originating at (x,y), precisely one in each direction.*

The line segments need not be of the same length. For each (x,y), let

$$\mathscr{B}_{(x,y)} = \{B_{(x,y)} : B_{(x,y)} \text{ is a radial ball centered at } (x,y)\}$$

Show that the collection, $\mathscr{B} = \cup\{\mathscr{B}_{(x,y)} : (x,y) \in \mathbb{R}^2\}$, forms a valid base for a topology, τ^*, on $\mathbb{R}^2$. Then show that the topology, τ^*, generated by $\mathscr{B}$ is strictly stronger than the usual topology, τ.

Solution: First part: We *claim* that $\mathscr{B}$ satisfies the "open base property" and so generates a topology on $\mathbb{R}^2$. Clearly, $\mathbb{R}^2 = \cup\{B_{(x,y)} : (x,y) \in \mathbb{R}^2\}$. Suppose $(x,y) \in U \cap V$ where U, $V \in \mathscr{B}_{(x,y)}$. Let K_U be a line segment in a particular direction originating at (x,y) such that $K_U \subset U$ and K_V be a line segment originating at (x,y) pointing in the same direction as K_U but $K_V \subset V$. Let $K_{U\cap V} = K_U \cap K_V$. Then $K_{U\cap V}$ is an open line segment originating at (x,y) which is contained in $U \cap V$. This can be repeated for all open lines in all directions originating at (x,y) to obtain a radial ball, $M_{(x,y)}$. So there exists $M_{(x,y)} \in \mathscr{B}_{(x,y)}$ such that

$$(x,y) \in M_{(x,y)} \subseteq U \cap V$$

So $\mathscr{B}$ satisfies the open base property.

Hence, by Theorem 5.4, $\mathscr{B}$ is an open base which generates a topology on $\mathbb{R}^2$, say τ^*. We are done with the first part of the question.

Second part: We *claim* that the usual topology, τ, is contained in τ^*. Suppose $(x, y) \in U$ where U is an open base element of τ centered at (x, y). Then $U = B_\varepsilon(x, y)$ for some ε. Let $(a, b) \in B_\varepsilon(x, y)$. Then there exists $\delta > 0$ such that $B_\delta(a, b) \subseteq B_\varepsilon(x, y)$. Let V be a set such that all open line segments originating at (a, b) are of length less than $\delta/2$. Then $V \in \mathscr{B}$ and $V \subseteq B_\delta(a, b) \subseteq B_\varepsilon(x, y)$. So $U \in \tau^*$. Then $\tau \subseteq \tau^*$, as claimed.

We *claim* that τ^* is strictly stronger than τ. To show this, it suffices to show that τ^* contains an element which does not belong to τ. Consider the sets $U = B_1(0, 1)$ and $V = B_1(0, -1)$.[4] Then, if $L = \{(x, 0) : x \in \mathbb{R}, |x| \leq k\}$ the set,

$$W = U \cup L \cup V$$

representing the two balls U and V with a straight horizontal line through $(0, 0)$ added to them does not belong to τ. However, the set $M(0, 0)$ which has rays emanating from $(0, 0)$, in all directions, each of which is contained in W is an open neighborhood of $(0, 0)$ in τ^*. So τ is a proper subset of τ^*, as claimed.

5.4 The subbase of a topology.

We have seen that, when given an arbitrary set S, a subfamily of $\mathscr{P}(S)$ which satisfies the "base property" can be used to generate a topology on S. We shall soon see that any subfamily family of $\mathscr{P}(S)$ can be used to generate a base which, in turn, will generate a topology, τ, on S. We will refer to such a family as a "subbase" for τ. Even if this may, at first, appear to be a very convenient way to generate topologies on a set, it does make it more difficult to predict, from a subbase, what the topology generated by such a collection of sets will be like. So, for this convenience, there is a price to pay. We know that we can obtain a topology from a collection of sets which satisfies the "base property" in a single step: base → τ. But to obtain a topology, τ, from a subbase requires two steps:

$$\text{subbase} \rightarrow \text{base} \rightarrow \tau$$

But the notion of a subbase is nevertheless intriguing enough to investigate since, if a space is already equipped with a topology, τ, it can be useful to understand what kind of subbase generates τ. It may allow

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[4] The set $B_1(0, 1)$ refers to the open ball center $(0, 1)$ and radius 1, while $B_1(0, -1)$ refers to the open ball center $(0, -1)$ with radius 1.

us to dig deeper into the background of τ to better see why τ satisfies certain properties.

Definition 5.7 Let (S, τ) be a topological space. A *subbase for the topology* τ is a non-empty subfamily, $\mathscr{S}$, of τ such that the set, $\mathscr{B}$, defined as

$$\mathscr{B} = \{B : B = \cap\{U : U \in \mathscr{F}\} \text{ where } \mathscr{F} \text{ is a finite subset of } \mathscr{S}\}$$

forms a base for the open sets in τ.

The definition states that, if a base of τ can be obtained from the set of all finite intersections of the elements of a collection, $\mathscr{S}$, then $\mathscr{S}$, is called a subbase of τ. This of course means that every element of a subbase of τ belongs to τ and so is open.

In the following two examples we are given a particular topology on a set. For each topological space we identify a subbase for the given topology.

Example 4. *A subbase for the usual topology on* $\mathbb{R}$. Consider the sets

$$\begin{aligned} (a, \infty) &= \{x \in \mathbb{R} : a < x\} \\ (-\infty, b) &= \{x \in \mathbb{R} : x < b\} \end{aligned}$$

and the family

$$\mathscr{S} = \{(a, \infty) : a \in \mathbb{R}\} \cup \{(-\infty, b) : b \in \mathbb{R}\}$$

We see that $\mathscr{S}$ forms a subbase for the usual topology τ on $\mathbb{R}$ since the set of all finite intersections of elements of $\mathscr{S}$ form the set, $\mathscr{B} = \{(a, b) : a < b\}$, known to be a base of τ.

Example 5. Consider $(\mathbb{N}, \tau_d)$ where τ_d is the *discrete topology* (see the definition on page 39). Then $\tau_d = \mathscr{P}(\mathbb{N})$.

The set of all singleton sets, $\mathscr{B} = \{\{n\} : n \in \mathbb{N}\}$, forms a base for τ_d since every subset of $\mathbb{N}$ is the union of elements from some subset $\mathscr{C} \subseteq \mathscr{B}$. If

$$\begin{aligned} N_a &= \{n \in \mathbb{N} : n \leq a\} \\ M_b &= \{n \in \mathbb{N} : n \geq b\} \end{aligned}$$

and

$$\mathscr{S} = \{N_a : a \in \mathbb{N}\} \cup \{M_b : b \in \mathbb{N}\}$$

then $\mathscr{B}$ is a subset of the family of finite intersections of elements of $\mathscr{S}$. So $\mathscr{S}$ is a subbase for the topology τ_d on $\mathbb{N}$.

5.5 A subbase as a generator of a topology.

We now show how *any* non-empty subset, $\mathscr{S}$, of $\mathscr{P}(S)$ will generate a topology, $\tau_{\mathscr{S}}$ on S. This topology, will have $\mathscr{S}$ as "subbase".

Suppose $\{\tau_j : j \in I\}$ represents the class of all topologies on S. Suppose S is a set and $\mathscr{S}$ is a non-empty subset of $\mathscr{P}(S)$. Let,

$$\mathscr{J} = \{\tau_j : j \in I, \mathscr{S} \subseteq \tau_j\}$$

denote *all* topologies on S which contain $\mathscr{S}$. The set $\mathscr{J}$ is non-empty since it at least contains the discrete topology, $\tau_d = \mathscr{P}(S)$, which, by definition, contains $\mathscr{S}$. Since the family of all topologies on a set is closed under intersections (see page 39), (but not necessarily under unions), then the family

$$\tau_{\mathscr{S}} = \cap\{\tau_j : j \in I,\, \mathscr{S} \subseteq \tau_j\}$$

is also a topology on S which contains all elements of $\mathscr{S}$. In fact, it is easily verified that $\tau_{\mathscr{S}}$ *is the smallest possible topology on* S *which contains* $\mathscr{S}$.

In the following theorem we show that $\mathscr{S}$ is a subbase for the constructed topology, $\tau_{\mathscr{S}}$, on S.

Theorem 5.8 Let S be a non-empty set and $\mathscr{S} \subseteq \mathscr{P}(S)$. Suppose

$$\tau_{\mathscr{S}} = \cap\{\tau_j : \tau_j \text{ is a topology on } S,\, \mathscr{S} \subseteq \tau_j\}$$

Then $\mathscr{S}$ is a subbase for $\tau_{\mathscr{S}}$ on S which contains $\mathscr{S}$.

Proof: Given: $\mathscr{S} \subseteq \mathscr{P}(S)$ and $\tau_{\mathscr{S}} = \cap\{\tau_j : j \in I, \tau_j$ is a topology on S, $\mathscr{S} \subseteq \tau_j\}$.

We have already seen that $\tau_{\mathscr{S}}$ is the smallest topology on S with contains $\mathscr{S}$. Let

$$\mathscr{B} = \{B : B = \cap_{M \in \mathscr{F}} M \text{ where } \mathscr{F} \text{ is a finite subset of } \mathscr{S}\}$$

Claim: We claim that $\mathscr{B}$ is a basis for a topology τ on S.
Note that, the empty set, $\varnothing$, is a finite subset of $\mathscr{S}$. Then

$$\cap_{M\in\varnothing} M = \{x \in S : x \in M \text{ for every } M \in \varnothing\} = S$$

So $S \in \mathscr{B}$.

Let $A, D \in \mathscr{B}$ and $\mathscr{A}$ and $\mathscr{D}$ be finite subfamilies of $\mathscr{S}$ such that $A = \cap_{M\in\mathscr{A}} M$ and $D = \cap_{M\in\mathscr{D}} M$. Then $\mathscr{A} \cup \mathscr{D}$ is a finite subfamily of $\mathscr{S}$. Suppose $x \in A \cap D$. Then

$$x = (\cap_{M\in\mathscr{A}} M) \cap (\cap_{M\in\mathscr{D}} M) = \cap_{M\in\mathscr{A}\cup\mathscr{D}} M = A \cap D$$

If $E = \cap_{M\in\mathscr{A}\cup\mathscr{D}} M$, $x \in E \subseteq A \cap D$.

Then $\mathscr{B}$ is a base for *some* topology τ on S, as claimed. Also, $\mathscr{S}$ is a subbase for τ.

Since $\tau_{\mathscr{S}}$ is the smallest topology which contains $\mathscr{S}$, then $\tau_{\mathscr{S}} \subseteq \tau$. On the other hand, if $U \in \tau$, then U is the union of elements of $\mathscr{B} \subseteq \tau_{\mathscr{S}}$. So $U \in \tau_{\mathscr{S}}$. We conclude that $\tau \subseteq \tau_{\mathscr{S}}$.

Then $\tau_{\mathscr{S}} = \tau$ and so $\mathscr{S}$ is a subbase of $\tau_{\mathscr{S}}$.

The above theorem guarantees that any non-empty subset of $\mathscr{P}(S)$ is a subbase for some topology on S. We provide the following examples where this principle is applied.

Example 6. Consider the sets

$$\begin{array}{rcl} X_a & = & \{x \in \mathbb{R} : a < x\} \\ Y_b & = & \{x \in \mathbb{R} : x \leq b\} \end{array}$$

and the family $\mathscr{S} = \{X_a : a \in \mathbb{R}\} \cup \{Y_b : b \in \mathbb{R}\}$. We see that the intersections of finite subsets of $\mathscr{S}$ are either $\varnothing$ or can be of the form $(-\infty, b]$, (a, ∞) or $(a, b]$. These subsets form a subbase for a topology on $\mathbb{R}$. Note that $(-\infty, b) = \cup\{(a, b-\varepsilon] : a < b-\varepsilon, \varepsilon \in (0,1)\}$ and $(c, \infty) = \cup\{(c, b] : b > c\}$. Then the set

$$\mathscr{B} = \{(a, b] : a < b\} \cup \{\varnothing\}$$

is sufficient to form a base for a topology, $\tau_{\mathscr{S}}$, on $\mathbb{R}$ generated by the subbase $\mathscr{S}$. This topology is referred to as the

“*upper limit topology on* $\mathbb{R}$”

In the same line of thought, we can also topologize $\mathbb{R}$ with the following similar family of subsets of $\mathbb{R}$,

$$\mathscr{B} = \{[a, b) : a < b\} \cup \{\varnothing\}$$

The set $\mathscr{B}$ is easily seen to satisfy the "base property" and so generates a topology on $\mathbb{R}$. We refer to this topology as the *lower limit topology* and denote it as, τ_S. When $\mathbb{R}$ is equipped with the lower limit topology, we refer to $(\mathbb{R}, \tau_S)$ as the

Sorgenfrey line

Sets of the form $[a, b)$ in the Sorgenfrey line are also seen to be closed with respect to τ_S (since $\mathbb{R}\backslash[a, b)$ is the union of the two open subsets $\cup\{[b, c) : c > b\}$ and $\cup\{[c, a - \varepsilon) : \varepsilon \in (0, 1), c < a - \varepsilon\}$). So...

the Sorgenfrey line has an open base made of clopen sets.

We also easily see that every singleton set $\{b\}$ is closed with respect to τ_S and so every interval of the form $[a, b]$ in $(\mathbb{R}, \tau_S)$ is closed. So $\mathbb{R}$ equipped with the usual topology is strictly weaker then the topology on the Sorgenfrey line.

Of course, we don't construct a topology on a space, S, from an arbitrary subset of S just to let it sit there on the table. There is still some work to do. We must determine some of its properties. This will come around later in the text.

Example 7. Consider the set, $\mathscr{S} = \{[a, b] : a < b\}$, of all non-empty closed and bounded intervals in $\mathscr{P}(\mathbb{R})$.

We see that $\mathscr{S}$ does not satisfy the "base property" (since there does not exist $[x, y]$ in $\mathscr{S}$ such that $[x, y] \subseteq [a, b] \cap [b, c]$) and so cannot form a base for a topology on $\mathbb{R}$.

However, the theorem guarantees that $\mathscr{S}$ is a subbase for some topology on $\mathbb{R}$. Describe this topology.

Solution: We see that non-empty finite intersections of elements of $\mathscr{S}$ are of the form $[c, d]$ where $c \leq d$. In particular, $[u, x] \cap [x, v] = \{x\}$ is an open base element for all $x \in \mathbb{R}$. So the subbase $\mathscr{S}$ will generate an open base, $\mathscr{B} = \{\{x\} : x \in \mathbb{R}\}$, for the discrete topology, τ_d.

We see that the subbase, $\mathscr{S}$, holds more sets than is really required to generate τ_d. We can reduce its size to

$$\mathscr{S}^* = \{[x - 1, x] : x \in \mathbb{R}\} \cup \{[x, x + 1] : x \in \mathbb{R}\}$$

Since, for each $x \in \mathbb{R}$, $[x-1, x] \cap [x, x+1] = \{x\}$, $\mathscr{S}^*$ still generates τ_d.

Example 8. Consider two topological spaces (S, τ_S) and (T, τ_T). Using the sets S and T we can construct a new set, the Cartesian product $S \times T$, defined as $S \times T = \{(x, y) : x \in S, y \in T\}$. We can topologize the set $S \times T$ by defining a suitable subbase.

We will proceed as follows. Define the two projection functions $\pi_S : S \times T \to S$ and $\pi_T : S \times T \to T$ as $\pi_S(x, y) = x$ and $\pi_T(x, y) = y$, respectively. We will define as subbase for $S \times T$

$$\mathscr{S} = \{\pi_S^{-1}(U) : U \in \tau_S\} \cup \{\pi_T^{-1}(V) : V \in \tau_T\}$$

where $\pi_S^{-1}(U) = U \times T$ and $\pi_T^{-1}(V) = S \times V$. By referring to the principle $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$ (verify this!), the basis $\mathscr{B}$ induced by this subbase is of the form

$$\mathscr{B} = \{U \times V : U \in \tau_S, V \in \tau_T\}$$

The topology on $S \times T$ whose base is $\mathscr{B}$ is called the *product topology on* $S \times T$. Such topological spaces created by combining known topological spaces are important and so will be discussed in depth in the next few chapters.

Equivalent bases. Two bases $\mathscr{B}$ and $\mathscr{B}^*$ are said to be *equivalent bases* for a particular set S if they generate the same topology.

Note that the usual basis for $\mathbb{R}$ and the basis for the *upper limit topology* described in the example above are not equivalent bases since $(a, b]$ does not belong to the usual topology on $\mathbb{R}$. However, since

$$(a, b) = \cup\{(a, b - 1/n] : n = 1, 2, 3, \ldots\} \in \tau_{\mathscr{S}}$$

, then every base element for the usual topology belongs to $\tau_{\mathscr{S}}$ and so … "the usual topology is weaker than $\tau_{\mathscr{S}}$".

In the following example, we illustrate a different topology, τ_{scat} on $\mathbb{R}$. The topological space, $(\mathbb{R}, \tau_{\text{scat}})$, is known as the "*scattered line*" or the "*discrete irrational extension of* $\mathbb{R}$".

Example 9. Consider the real line, $\mathbb{R}$. Let τ denote the usual topology on $\mathbb{R}$ and $\mathbb{J}$ denote the subset of all irrationals. Let

$$\tau_{\text{scat}} = \{U \cup V : U \in \tau \text{ and } V \subseteq \mathbb{J}\}$$

Verify that τ_{scat} is indeed a topology on $\mathbb{R}$. Find a base for this topology.

Solution. O1. Since $\varnothing \in \tau$ and $\varnothing \subseteq \mathbb{J}$, $\varnothing = \varnothing \cup \varnothing \in \tau_{\text{scat}}$. So $\varnothing \in \tau_{\text{scat}}$. Since $\mathbb{R} = \mathbb{R} \cup \mathbb{J}$, then $\mathbb{R} \in \tau_{\text{scat}}$.

O2. Let $\mathscr{A} = \{U_i \cup V_i : i \in I\}$ be a subfamily of τ_{scat}. Then

$$\begin{aligned} \cup\mathscr{A} &= \cup\{U_i \cup V_i : i \in I\} \\ &= \bigcup_{i\in I} U_i \ \cup \ \bigcup_{i\in I} V_i \end{aligned}$$

where $\bigcup_{i\in I} U_i \in \tau$ and $\bigcup_{i\in I} V_i \subseteq \mathbb{J}$. So $\cup\mathscr{A} \in \tau_{\text{scat}}$.

O3. Let $\mathscr{D} = \{U_i \cup V_i : i = 1 \text{ to } n\}$ be a finite subfamily of τ_{scat}. Then

$$(U_1 \cup V_1) \cap (U_2 \cup V_2) \quad = \quad (U_1 \cap U_2) \ \cup \ [(U_1 \cap V_2) \cup (V_1 \cap (U_2 \cup V_2))]$$

The right-hand side is an element of τ_{scat} (where $U_1 \cap U_2 \in \tau$ and $(U_1 \cap V_2) \cup (V_1 \cap (U_2 \cup V_2)) \subseteq \mathbb{J}$). Proceeding by finite induction on n, we conclude O3 is satisfied.

So τ_{scat} is indeed a topology on $\mathbb{R}$.

Suppose $A \in \tau_{\text{scat}}$ contains the rational number q. Then $A = U \cup V$ for some $U \in \tau$ and $V \subseteq \mathbb{J}$. Since q is not irrational, $q \in U$. Then there exists $\varepsilon > 0$ such that $q \in (q - \varepsilon, q + \varepsilon) \subseteq U \subseteq A$. So the family $\{(q - \varepsilon, q + \varepsilon) : \varepsilon > 0\}$ forms a neighborhood base for q in τ_{scat}.
On the other hand, if $r \in \mathbb{J}$, and $A \in \tau_{\text{scat}}$ contains r, then $\{r\} = \varnothing \cup \{r\} \in \tau_{\text{scat}}$. So $\{\{r\} : r \in \mathbb{J}\}$ forms a neighborhood base of r.

Then

$$\{\{r\} : r \in \mathbb{J}\} \cup \{(q - \varepsilon, q + \varepsilon) : \varepsilon > 0, q \in \mathbb{Q}\}$$

forms a base for the scattered line.

5.6 Topic: Spaces with countable bases.

Some properties of a topological space, S, may depend on the cardinality of an open base on S. We will see that some spaces have a countable open base, while others simply have a countable neighborhood base *at each point.* It will be useful to discuss these notions as soon as possible in the text.

Definition 5.9 Let (S, τ) be a topological space. The topological space, S, is said to be

“*first countable*”

if and only if every point, $x \in S$, has a countable neighborhood base, $\mathscr{B}_x$. The topological space, S, is said to be

"*second countable*"

if and only if the smallest of its open bases is a countable base.

The three countability properties of a space. We have discussed a countability property which is indirectly related to the topology of a space before. Recall (from Definition 4.10) that a space is

"*separable*"

if it has a countable dense subset. The three properties, *separable*, *first countable* and *second countable* are often referred to as "the countability properties" of a topological space. When exploring an unfamiliar topological space we usually try to determine immediately which of these three countability properties (if any) are satisfied in this space. We begin by discussing in the ways in which these properties relate to each other.

Theorem 5.10 A second countable topological space is a first countable space.

Proof: Suppose (S, τ) is a second countable space. Then S has a countable base, $\mathscr{B}$. Then, for each $x \in S$, $\mathscr{B}_x = \{B \in \mathscr{B} : x \in B\}$ is a neighborhood base at x. Since $\mathscr{B}$ is countable and $\mathscr{B}_x \subseteq \mathscr{B}$, $\mathscr{B}_x$ cannot be uncountable. So (S, τ) is first countable.

The class of all first countable spaces is a subclass of all second countable spaces. However, there are some uncountably large topological spaces which are first countable but not second countable. We illustrate such a space in the following example.

Example 10. Suppose τ_S represents the Sorgenfrey topology on $\mathbb{R}$. (It is also called the *lower limit topology* on $\mathbb{R}$; see example on page 93). Recall that the set $\mathscr{B} = \{[x, y) : x, y \in \mathbb{R}\}$ is an open base for $(\mathbb{R}, \tau_S)$. Show that $(\mathbb{R}, \tau_S)$ is first countable but not second countable. Also verify that the Sorgenfrey line is separable.

Solution: Since, for each $x \in \mathbb{R}$,

$$\mathscr{B}_x = \{[x, x + 1/n) : n \in \mathbb{N} \backslash \{0\}\}$$

is a countable neighborhood base at x. So $(\mathbb{R}, \tau_S)$ is first countable.

The space, $(\mathbb{R}, \tau_S)$ is not second countable: Suppose $\mathscr{B} = \{[a_n, b_n) : n \in \mathbb{N}\}$ is a countable set of basic elements. Then, since $\mathbb{R}$ is uncountable, there exists $z \in \mathbb{R}$ such that, for all $n \in \mathbb{N} \backslash \{0\}$, $z \neq a_n$. Then, for any $b > z$, $[z, b)$ is not the union of a subfamily of $\mathscr{B}$. So $(\mathbb{R}, \tau_S)$ does not have a countable base for open set.

So the Sorgenfrey line, $(\mathbb{R}, \tau_S)$, is first countable but not second countable. As required

To show that the Sorgenfrey line, $(\mathbb{R}, \tau_S)$, is separable we must show that it has a countable dense subset. Since, for any $z \in \mathbb{R}$, the basic open neighborhood $[z, b)$ of z contains a rational number, $\mathbb{Q}$ is a dense subset of $(\mathbb{R}, \tau_S)$. Then $(\mathbb{R}, \tau_S)$ is separable.

Metrizable spaces turn out to be first countable spaces. That is, every point of a metrizable space has a countable neighborhood base. We have the tools needed to prove this immediately.

Theorem 5.11 Any metrizable space is first countable.

Proof: Suppose (S, τ) is a metrizable space whose open sets are generated by the metric ρ. Then the set

$$\mathscr{B}_x = \{B_{1/n}(x) : n \in \mathbb{N}, n > 0\}$$

(an open ball center x with radius $1/n$) forms a countable neighborhood base at x.

Hence S is a first countable space.

Example 11. Since $\mathbb{R}$ (equipped with usual topology) is metrizable, then it is first countable.

Example 12. Verify that the space, $\mathbb{R}$, when equipped with the usual topology, is second countable.

Solution: We are required to show $\mathbb{R}$ has a countable open base.

Suppose U is an open subset of $\mathbb{R}$ and $x \in U$. Then there exists an open interval, $B_\varepsilon(x) = (x - \varepsilon, x + \varepsilon)$, such that $x \in B_\varepsilon(x) \subseteq U$.

We consider two cases.

- If $x \in \mathbb{Q}$, then there exists an integer m such that $x \in B_{\frac{1}{m}}(x) \subseteq B_\varepsilon(x) \subseteq U$.
- If x is an irrational, we know that there is a sequence of rationals which converges to x, so we can find a rational y in $B_{\frac{1}{4m}}(x)$, such that
$$x \in B_{1/4m}(y) \subseteq B_\varepsilon(x)^5$$

We have just shown that $\mathscr{B} = \{B_m(y) : y, m \in \mathbb{Q}\}$ forms a base for the open sets in $\mathbb{R}$. Since $\mathbb{Q}$ is countable, then so is $\mathbb{Q} \times \mathbb{Q}^6$ so

$$|\mathscr{B}| = |\{B_m(y) : (y, m) \in \mathbb{Q} \times \mathbb{Q}\}| = |\mathbb{Q} \times \mathbb{Q}| = \aleph_0$$

We have shown that, when equipped with the usual topology, $\mathbb{R}$ has a countable base for open sets.

Example 13. Recall that the radial plane, $(\mathbb{R}^2, \tau_r)$, is equipped with what is called the *radial plane topology*, τ_r, (see the example on page 89). The open sets in the radial plane are defined as follows: the subset U is an open neighborhood of q if and only if $q \in U$, and U is the union of a set of open line segments, precisely one in each direction, each one originating at q. It is shown on page 89, that this is a valid topology and that it is strictly stronger than the usual topology on $\mathbb{R}^2$. Verify that the radial plane is *not* first countable.

Solution: Let $p \in \mathbb{R}^2$. Suppose $(\mathbb{R}^2, \tau_r)$ is first countable. Then p has a countable open neighborhood base,

$$\mathscr{B}_p = \{B_i : i \in \mathbb{N}\backslash\{0\}\}$$

---

[5] Choose integer m so that $1/m < \varepsilon$. Choose rational y such that $y \in B_{1/4m}(x) \subset B_\varepsilon(x)$. If $u \in B_{1/4m}(y)$, then

$$\begin{aligned} |x-u| &= |x-y+y-u| \\ &\leq |x-y|+|y-u| \\ &< 1/4m + 1/4m \\ &= 1/2m < 1/m < \varepsilon \end{aligned}$$

So $x \in B_{1/4m}(y) \subseteq B_\varepsilon(x) \subset U$.

[6†] Since $\mathbb{Q}$ is countable, then so is $\mathbb{Q} \times \mathbb{Q}$. See R. André, *Set theory: An introduction to Axiomatic Reasoning* on cardinalities.

We will construct an element, V, of τ_r which contains p but does not contain any element of $\mathscr{B}_p$.

For each radial set, $B_i \in \mathscr{B}_p$, let r_i be the length of the longest ray originating at p which appears in B_i and $|r_i|$ denote its length. We then inductively construct a sequence of rays, $R = \{r_i^* : i \in \mathbb{N}\backslash\{0\}\}$ each originating at p in the direction of r_i and of length

$$|r_i^*| = |r_i/2^i|$$

Since we have only countably many rays the set,

$$\{r_i^* : i \in \mathbb{N}\backslash\{0\}\} \cup \{p\}$$

is not yet a complete radial set centered at p. We will add uncountably many more rays of arbitrary length to the ones we have to obtain a complete radial open set, V, centered at p. No open neighborhood B_i can be contained in this V (since the rays become arbitrarily small, shrinking down to p). So $\mathscr{B}_p$ cannot be an open neighborhood base for p. So p must have an uncountable open neighborhood base.

5.7 Topic: Relating the countable base property with the separable property.

Recall that in definition 4.10, we defined a *separable* topological space as being a space which has a countable dense subset. We saw, for example, that, since $\mathbb{Q}$ is a countable dense subset of the reals, $\mathbb{R}$ is separable.[7]

In the following theorem, we see that all second countable spaces are guaranteed to have a countable dense subset. The proof of this statement follows from a more general result (in part (a)) which states that, given any space S with open base, $\mathscr{B}$, S has a dense subset, D, whose cardinality, $|D|$, is less than or equal to the cardinality of $\mathscr{B}$.

Theorem 5.12 *"Second countable ⇒ separable" theorem.* Suppose $\mathscr{B}$ is an open base of the space (S, τ_S) where $|\mathscr{B}|$ denotes the cardinality of $\mathscr{B}$.

(a) There exists in S a dense subset, D such that $|D| \leq |\mathscr{B}|$.

(b) Any second countable topological space is a separable space.

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[7] Where $\mathbb{R}$ is equipped with the usual topology.

Proof: We are given that $\mathscr{B} = \{B_i : i \in I\}$ is a base such that, if any other base has cardinality J, $J \geq |I|$.

(a) For each $i \in I$, choose $x_i \in B_i$. (Choice!)[8] Let $D = \{x_i : i \in I\}$. Then $|D| \leq |\mathscr{B}|$.

We *claim* that D is dense in S: Let U be a non-empty open subset of S. Then, for $x \in U$ there exists $B_i \in \mathscr{B}$ such that $x \in B_i \subseteq U$. Then $x_i \in B_i \subseteq U$. Then $U \cap D \neq \varnothing$. So every open set in S intersects D. This means that D is dense in S, as *claimed.*

Then D is a dense subset of S such that $|D| \leq |I|$. Since I is the smallest indexing set for any open base, then D has a cardinality which is less than or equal to the cardinality of any base.

(b) It follows immediately from part (a) that "second countable property" implies the "separable property". For, if $|\mathscr{B}| = \aleph_0$, then D is a countable dense subset of S. So S is separable.

We have just shown that those spaces that have a countable base of open sets must have a countable dense subset. In general, the converse does not hold true. That is, there are spaces that have a countable dense subset which do not have a countable base of open sets. But if S is a metrizable space, then the converse holds true, as we shall now see.

Theorem 5.13 Separable metrizable spaces are second countable.

Proof: We are given that S is a metrizable separable topological space. Then there is a metric, ρ, on S such that the space, S, is equivalent to (S, ρ).

Since S is separable, then S has a countable dense subset, $D = \{x_i : i \in \mathbb{N}\backslash\{0\}\}$.

For each i and n in $\mathbb{N}\backslash\{0\}$ let $B_{(i,n)} = B_{1/n}(x_i)$, be an open ball center x_i and radius $1/n$. Consider

$$\mathscr{B} = \{B_{(i,n)} : i, n \in \mathbb{N}\backslash\{0\}\}$$

For $x \in S$, let U_x be an open neighborhood of x. Then there exists, $j \in \mathbb{N}\backslash\{0\}$ such that $B_{1/j}(x) \subseteq U_x$. Since D is dense in S, $B_{1/j}(x) \cap D$ is non-empty. Say,

$$x_k \in B_{\frac{1}{2j}}(x) \cap D$$

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[8] Existence theorems often (but not always) suggest an application of the Axiom of choice in the proof. Keep an eye open for it.

Then

$$x \in B_{\frac{1}{2j}}(x_k) \subseteq B_{\frac{1}{2j}}(x) \subseteq U_x$$

Then $\mathscr{B}$ is a countable base for open sets in S. So the separable metric space, S, is second countable.

The following diagram summarizes the above results.

$$\begin{tikzcd}
\text{metrizable + separable} \arrow[d, Rightarrow] & \text{metrizable} \arrow[d, Rightarrow] \\
\text{second countable} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{first countable} \\
\text{separable} &
\end{tikzcd}$$

Example 14. From the above statements, we have another proof that the set of all reals equipped with the usual topology is both separable and second countable.

We can also conclude from the theorem that...

the Sorgenfrey line is not a metrizable space.

Since we have shown in an example above that $(\mathbb{R}, \tau_S)$ is both first countable and separable. If it was also metrizable then it would have to be second countable. But we have proven that this is not the case.

5.8 Topic : Hereditary topological properties.

Some properties on spaces are carried over from the whole space to their subspaces, while others are not. Those properties that do are called "hereditary properties".

Definition 5.14 A topological property, say P, of a space (S, τ_S) is said to be a *hereditary topological property* provided every subspace, (T, τ_T), of S also has P.

Example 15. Show that metrizability is a hereditary property.

Solution: Suppose (S, τ) is metrizable. Then there exists a metric ρ such that (S, τ) and (S, ρ) have the same open sets. Suppose $T \subseteq S$ has the subspace topology and $\rho_t : T \times T \to \mathbb{R}$ is the subspace metric on T. Then (T, τ_t) and (T, ρ_t) have the same open sets and so T is metrizable. So subspaces of metrizable spaces are metrizable.

"First countable" is another example of a hereditary topological property.

However, the reader may want to verify that, if S is separable and V is an *open* subspace of S, then V is separable. But, in general, separability is not a hereditary property. (An example supporting this fact is found later in the book on page 526.[9])

We now show that the "second countable" property is hereditary.

Theorem 5.15 Suppose (S, τ_S) is a second countable topological space. Then any non-empty subspace of S is also second countable. So "second countable" is a hereditary property.

Proof: Suppose (S, τ_S) has a countable base $\mathscr{B} = \{B_i : i \in \mathbb{N}\}$. Suppose (T, τ) is a non-empty subspace of S. Let U be an open subset of T. Then there exists an open subset U^* of S such that $U = U^* \cap T$. Then there exists $N \subseteq \mathbb{N}$ such that $U^* = \cup\{B_i : i \in N\}$. Then $U = \cup\{B_i \cap T : i \in N\}$. So $\mathscr{B}_T = \{B_i \cap T : i \in \mathbb{N}\}$ is a countable basis of T. Hence T inherits the second countable property from its superset S.

"Separable metrizable" is hereditary.

It is immediately worth noting that the above results allow us to conclude that *subspaces of separable metrizable spaces are separable.* That is, the "separable metrizable" property is hereditary. To see this, simply note that, if T is a subspace of the a separable metrizable space S, then T is metrizable (by the above example). We claim that T is separable: From Theorem 5.13, the metrizable space, S, must be second countable. By Theorem 5.15, the second countable property is

---

[9] Where it is shown that $\beta\mathbb{N}\backslash\mathbb{N}$ is not separable even though $\beta\mathbb{N}$ is known to be separable.

hereditary. So T must be both metrizable and second countable. Since second countable spaces are separable (by Theorem 5.12), then T is a both metrizable and separable.

So, for example, should one want to argue that the irrationals, $\mathbb{J}$ (equipped with the usual topology), forms a separable space, it suffices to justify that $\mathbb{R}$ is both metrizable and second countable.

5.9 Topic : Ordinal space.

Ordinal numbers play a useful role in general topology. The particular topological properties of this "linearly well-ordered" set often serve as a rich source of counterexamples to certain conjectures. One reason being that it can be chosen to be as uncountably large as we want. A second reason is that it is "well-ordered". Uncountably large well-ordered sets exist with the assumption of the Well-ordering theorem, one of the least friendly forms – some might say "one of the least believable" – forms of the Axiom of choice. Recall that a linearly ordered set is said to be "well-ordered" if,

"*…every non-empty subset contains its least element.*"

Many of its properties can appear counter-intuitive, which is why some shy away from referring to them so as to not alienate some readers.

We state some of the most fundamental facts about ordinal numbers.

We remind ourselves of the definition. An *ordinal number* is a set whose elements are themselves sets.[10] We say that the set, α, is an ordinal number if it satisfies the two properties:

- If u, v are elements of α, they are themselves "sets of sets" such that either $u \in v$, $v \in u$, or $u = v$
- If $u \in v$, and $v \in \alpha$, then $u \in \alpha$.

The elements of an ordinal number, α, are linearly ordered by "$\in$ or $=$". The first countable ordinal numbers are in fact the natural numbers. Each ordinal is the set whose elements are all of its predecessors.

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[10] The family of *all* ordinal numbers is too large to be called a "set". So we refer to it as the *class* of all ordinal numbers.

$$\begin{array}{lcl}
\varnothing & = & 0 \\
\{\varnothing\} & = & \{0\} = 1 \\
\{\varnothing, \{\varnothing\}\} & = & \{0,1\} = 2 \\
\{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\} & = & \{0,1,2\} = 3 \\
\{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}, \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}\} & = & \{0,1,2,3\} = 4 \\
\quad\vdots & \vdots & \\
n \cup \{n\} & = & \{0,1,2,\ldots,n\} = n+1 \\
\quad\vdots & \vdots & \\
\{0,1,2,3,\ldots\} & = & \omega \\
\omega \cup \{\omega\} & = & \omega + 1 \\
\quad\vdots & \vdots &
\end{array}$$

See that $n+1 = \{0,1,2,3,\ldots,n\}$ is both a subset of $n+2$ and an ordinal contained in the next ordinal, $n+2 = \{0,1,2,3,\ldots,n,n+1\}$. Also, see that any $n+1$ is both a subset of ω and an element of ω (where ω is the first countably infinite ordinal).

All elements of an ordinal are themselves ordinals (a statement which requires proof). Those ordinals that do not contain a maximal ordinal (such as ω, for example) are referred to as *limit ordinals.* These can also be recognized by the fact that limit ordinals don't have an immediate predecessor. The *limit* ordinal, ω, can be expressed as a half-open interval,

$$\omega = [0,\omega) = \{\text{ordinal } \alpha : \alpha < \omega\}$$

where "$\alpha < \omega$" is interpreted as $\alpha \in \omega$. A non-limit ordinal, $\beta + 1 = \{0,1,2,\ldots,\beta\}$ can be expressed as an interval

$$\beta + 1 = [0,\beta] = [0,\beta+1) = \{\alpha : \alpha \leq \beta\}$$

where "$\alpha \leq \beta$" is interpreted as "$\alpha \in \omega$ or $\alpha = \beta$".

In the case of non-limit ordinals, $\beta + 1 = [0,\beta]$,

$$\begin{array}{rcl}
\cup\,(\beta+1) = \cup[0,\beta] & = & \cup\{\alpha : \alpha \text{ “}\in_{\text{or}}\text{=” } \beta\} = \beta \\
\sup\,(\beta+1) = \sup\,[0,\beta] & = & \sup\{\alpha : \alpha \text{ “}\in_{\text{or}}\text{=” } \beta\} = \beta
\end{array}$$

Limit ordinals are quite different in nature. We can recognize non-zero limit ordinals by the following characterization. The following three statements are equivalent:

- The ordinal, $\gamma = [0,\gamma) = \{\alpha : \alpha \in \gamma\}$, is a limit ordinal.
- For the ordinal γ, $\cup\gamma = \cup[0,\gamma) = \gamma$.
- For the ordinal γ, $\sup\gamma = \sup[0,\gamma) = \gamma$.

From this characterization, we can eventually argue (this is not immediate) that the union of all countable ordinals is the "least uncountable ordinal" or "first uncountable ordinal" denoted as, ω_1. The ordinal $\omega_1 \cup \{\omega_1\} = \omega_1 + 1$ is its successor.

5.10 Topic : Topologizing an ordinal set.

We will topologize an initial segment of ordinals, $S = [0, \omega_\alpha]$, (either with an open or closed right end) by defining an appropriate subbase, $\mathscr{S}$, which will generate a collection, $\mathscr{B}$, satisfying the "base property". The collection, $\mathscr{B}$, will, in turn, generate a topology on S.

Definition 5.16 Let ω_γ be an ordinal and $S = [0, \omega_\gamma] = \{\text{ordinals } \alpha : \alpha \leq \omega_\gamma\}$.

Suppose β and μ are both ordinals which belong to S. Let

$$\begin{aligned} S_\mu &= (\mu, \omega_\gamma] = \{\alpha \in S : \alpha > \mu\} \\ S_\beta &= [0, \beta) = \{\alpha \in S : \alpha < \beta\} \end{aligned}$$

The standard subbase of S is defined as,

$$\mathscr{S} = \{S_\mu : \mu \in S\} \cup \{S_\beta : \beta \in S\} \quad = \quad \{(\mu, \omega_\gamma] : \mu \in S\} \cup \{[0, \beta) : \beta \in S\}$$

This subbase will generate a base, $\mathscr{B}$, which in turn will generate the topology, τ_ω, of S.

When the space S is equipped with the topology τ_ω, (S, τ_ω), is referred to as an

"*ordinal space*"

The topology that is generated by this subbase is called the

"*interval topology on the set of ordinals*"

A few facts about an ordinal space. When we say "ordinal space" we mean a set of ordinals with the topology generated by the described subbase $\mathscr{S}$. But the best way to memorize the topology of the ordinal space is to remember what the elements of its base for open sets look like. Remember that there are two types of ordinals: limit ordinals and successor (non-limit) ordinals. Every ordinal number, α, without exception, has an immediate successor, $\alpha + 1$, by definition. Some ordinals, γ, have an immediate predecessor, say β, provided

$$\gamma = \beta + 1$$

In this case, $\sup\,[0, \gamma) = \sup\,[0, \beta + 1) = \sup\,[0, \beta] = \beta$.

While some ordinals, μ don't have an immediate predecessor (limit ordinal). In this case,

$$\sup\,\{\delta : \delta < \mu\} = \sup\,[0, \mu) = \mu$$

So, when we consider the intersection of two elements, $(\mu, \omega_\gamma]$ and $[0, \beta)$, of the subbase, $\mathscr{S}$, we get the open-ended interval (μ, β). At this point, we see only two possible cases for β:

- Case 1 : Suppose β has an immediate predecessor, say γ (because $\beta = \gamma + 1$). In this case, we can express the open base element, (μ, β), as the half-open interval, $(\mu, \gamma]$.
- Case 2 : Suppose β doesn't have an immediate predecessor. Then $\beta = \{\delta : \delta < \beta\}$. In this case, we must leave (μ, β) as is.

Another interesting fact to retain: The first uncountable ordinal, ω_1, cannot be reached (from below) by any sequence of ordinals. Another way of expressing this fact is to say

> "*No countable subset of* ω_1 *is cofinal.*"[11]

To see this, suppose S is a cofinal subset of ω_1. Then $\cup\{[0, \alpha) : \alpha \in S\} = \omega_1$. Since each ordinal α in S is countable, if S is countable then ω_1 would have to be countable set, a contradiction.

Example 16. Show that the first uncountable limit ordinal, $[0, \omega_1)$ is dense in $\omega_1 + 1$ (with respect to the ordinal topology).

Solution: Recall that ω_1 is a limit ordinal equal to $[0, \omega_1)$, the set of all countable ordinals. The set $\omega_1 = [0, \omega_1)$ does not contain the ordinal ω_1. Then $\omega_1 \cup \{\omega_1\} = \{0, \ldots, \omega_1\} = [0, \omega_1] = \omega_1 + 1$. We are required to show that $[0, \omega_1)$ is a dense proper subset of $[0, \omega_1]$.

Let (μ, β) be a basic open neighborhood of ω_1. Then $\mu < \omega_1 < \beta$. The expression $\mu < \omega_1$ is to be interpreted as $\mu \in \omega_1$. Then, since

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[11] A subset S of ω_1 is said to be cofinal if $S \cap (\alpha, \omega_1) \neq \varnothing$ for all $\alpha < \omega_1$.

ω_1 is a limit ordinal, there is an ordinal γ such that $\mu < \gamma < \omega_1$. So $(\mu, \beta) \cap \omega_1 \neq \varnothing$. Then $\omega_1 = [0, \omega_1)$ is dense in $\{0, \ldots, \omega_1\} = \omega_1 \cup \{\omega_1\} = \omega_1 + 1$.

Example 17. Suppose A and B are two *disjoint* closed subsets of the first uncountable limit ordinal, ω_1. Show that one of these two closed subsets of ω_1 must be a bounded subset. That is, at least one of these two subsets is not cofinal in ω_1.

Solution: Suppose both A and B are both cofinal and closed. We can then construct an *increasing* sequence of ordinals $\{\alpha_n : n \in \mathbb{N}\}$ such that $\{\alpha_0, \alpha_2, \alpha_4, \ldots\} \subseteq A$ and $\{\alpha_1, \alpha_3, \alpha_5, \ldots\} \subseteq B$. Then $\sup\{\alpha_n : n \in \mathbb{N}\} = \sup\{\alpha_{2n} : n \in \mathbb{N}\} = \sup\{\alpha_{2n+1} : n \in \mathbb{N}\} = \kappa \in [0, \omega_1)$. Since A and B are closed then $\sup\{\alpha_{2n} : n \in \mathbb{N}\} \in A$ and $\sup\{\alpha_{2n+1} : n \in \mathbb{N}\} \in B$. Then $\kappa \in A \cap B$ contradicting $A \cap B = \varnothing$.

5.11 Topic : On bases generated by regular open sets.

Suppose we are given a topological space (S, τ). Recall that, in 4.11, we defined:

> *An open subset, U, of S is called a* regular open *subset if it satisfies the property,*
>
> $$U = \text{int}_S \text{cl}_S U$$

In the expression, $\text{int}_S \text{cl}_S U$, the interior and closure are with respect to the topology τ on S. The symbol

$$\mathscr{R}o(S) = \{U \in \tau : U \text{ is regular open}\}$$

represents the set of all regular open subsets of S. So $\mathscr{R}o(S) \subseteq \tau$. But since $\mathscr{R}o(S)$ is not closed under unions (see the example on page 75), then it is not, by itself, a topology on S.

Clearly, $\varnothing$ and S belong to $\mathscr{R}o(S)$. On page 75, we showed that $\mathscr{R}o(S)$ is closed under finite intersections. Then, if $x \in S$ and $x \in A \cap B$ where $\{A, B\} \subseteq \mathscr{R}o(S)$, since $A \cap B \in \mathscr{R}o(S)$, then $\mathscr{R}o(S)$ satisfies the "base property". Then, by Theorem 5.4,

$$\mathscr{R}o(S) \text{ is an open base for some topology, say } \tau_s, \text{ on } S$$

The elements of $\mathscr{R}o(S)$ are all open with respect to τ but not all elements in τ_s are necessarily regular open in S. Furthermore, there may be some open sets in τ which are not unions of elements in $\mathscr{R}o(S)$ and so these are not in τ_s. So we have possibly distinct topologies, τ_s and τ, where,

$$\mathscr{R}o(S) \subseteq \tau_s \subseteq \tau$$

Note that the topology generated by $\mathscr{R}o(S)$ is weaker than τ, but under certain conditions, τ and τ_s may be equivalent topologies.

Definition 5.17 Let (S, τ) be a topological space and τ_s denote the topology whose base is $\mathscr{R}o(S)$. We have seen that τ_s and τ may be distinct topologies for the same underlying set, S.

If τ_s is a proper subset of τ, the topological space, (S, τ_s), is called the *semiregularization of S with respect to τ.*

If $\tau = \tau_s$, then we will call (S, τ) a *semiregular* topological space.[12] That is, a semiregular topological space is a space such that for all $U \in \tau$, U is the union of regular open sets.[13]

The semiregularization of, (S, τ), is akin to running τ through a strainer where only the elements of τ_s successfully make it through. The reader should verify that the set of real numbers, $\mathbb{R}$, equipped with the usual topology is an example of a semiregular topological space. Hence it is its own semiregularization. We will refer to various properties of semiregular spaces further on in the text.

Regular open sets as well as *regular closed sets* play an important role in the form of examples found in the chapter titled *The Stone space* in the last sections on this book.

5.12 Topic : Spaces with a base of clopen sets.

We now consider the set, $\mathscr{B}(S) = \{U \in \mathscr{P}(S) : U \text{ is clopen}\}$, of all clopen sets in the topological space, (S, τ). The set $\mathscr{B}(S)$ is never empty since $\varnothing$ and S are elements of that set. Furthermore, if U and V belong to $\mathscr{B}(S)$ and $x \in U \cap V$, given that $U \cap V$ also belongs to

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[12] We will specify later in this text that, in the literature, semiregular spaces are assumed to be Hausdorff. But for now this is irrelevant.

[13] The choice of the name "semiregular space" suggests that we will be introduced to a "regular space" at some point. This is the case. We will show in chapter 9 that "regular spaces" (to be defined in that chapter) are semiregular spaces, but there exist semiregular spaces which are not regular spaces.

$\mathscr{B}(S)$, then $\mathscr{B}(S)$ satisfies the "base property".

This means that $\mathscr{B}(S)$ forms a base for some topology, τ_b, on S. Since $\mathscr{B}(S) \subseteq \tau$, then the topology, τ_b, is weaker than τ, but may, under certain conditions, be equivalent to it.

Definition 5.18 Let (S, τ) be a topological space. If every point in S has a neighborhood base of clopen sets the we say that (S, τ) is a *zero-dimensional topological space.*

We have already witnessed an example of a zero-dimensional space. We have shown above that the Sorgenfrey line is generated by a base of clopen sets, and so is a zero-dimensional topological space.

Example 18. Consider the subspace, $(\mathbb{Q}, \tau)$, of rational numbers with the subspace topology inherited from $\mathbb{R}$, itself equipped with the usual topology. Verify that $(\mathbb{Q}, \tau)$ is a zero-dimensional topological space.

Solution: Consider $\mathscr{U} = \{(a, b) : a < b,\ a \text{ and } b \text{ irrationals}\} \subseteq \mathbb{R}$. It is easily verified that this forms a base for open sets in $\mathbb{R}$ equipped with the usual topology. Then

$$\mathscr{U}_Q = \{U \cap \mathbb{Q} : U \in \mathscr{U}\} = \{(a, b) \cap \mathbb{Q} :\ a \text{ and } b \text{ irrationals}\}$$

It is easily verified that each element of $\mathscr{U}_{\mathbb{Q}}$ is clopen in $(\mathbb{Q}, \tau)$ and that it forms an open base for $(\mathbb{Q}, \tau)$. Then $\mathscr{U}_Q$ generates the topology, τ. By definition, $(\mathbb{Q}, \tau)$ is a zero-dimensional topological space.

This example confirms that there are non-discrete zero-dimensional spaces.

5.13 Topic : A topology on $\mathbb{Z}$, which is neither discrete nor indiscrete.

We now consider a topology on $\mathbb{Z}$ which is neither the discrete topology nor the indiscrete topology in an example which may present a bit more of a challenge to the reader.[14] The example has a number theoretic flavor to it, so students of number theory may find the problem easier to solve. But, in essence, it is still a topology problem.

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[14] "If there is no challenge, then where's the fun?"

Example 19. For each $k \in \mathbb{Z}$ and $n \in \mathbb{N}\backslash\{0\}$, let

$$B(n,k) = \{y \in \mathbb{Z} : y = mn + k, m \in \mathbb{Z}\}$$

For example,

$$B(5,2) = \{\ldots, (-2)5+2, (-1)5+2, (0)5+2, (1)5+2, (2)5+2, \ldots\}$$

Let $\mathscr{B} = \{B(n,k) : k \in \mathbb{Z}, n \in \mathbb{N}\backslash\{0\}\}$. Show that $\mathscr{B}$ forms an open base for some topology on $\mathbb{Z}$.

The solution is deferred to the Appendix B.

It is a good idea to attempt to solve the problem before glancing at the proposed solution in the appendix.

Example 20. *The ψ-space.*

a) Show that $\mathbb{N}$ contains an infinite family of infinite subsets such that any pair of sets in this family has a finite intersection.

b) Let $\overline{\mathscr{F}}$ denote the collection of all subsets $\mathscr{B}$ of $(\mathscr{P}(\mathbb{N}), \subseteq)$ such that every element of $\mathscr{B}$ is infinite and every pair F and G in $\mathscr{B}$, $F \cap G$ is finite. Show that the family $\overline{\mathscr{F}}$ has a maximal element, $\mathscr{M}$, with respect to "$\subseteq$".

c) Let $\psi = \mathbb{N} \cup \mathscr{M}$. Then

$$\psi \subseteq \mathbb{N} \cup \mathscr{P}(\mathbb{N})$$

Let

$$\mathscr{E} = \{\, \{D\} \cup D\backslash F : D \in \mathscr{M}, F \text{ is a finite subset of } D\,\}$$

Note that, if $D \in \mathscr{M}$, $D \subseteq \mathbb{N}$, so $\{D\} \in \mathscr{P}(\mathbb{N})$. Since $D\backslash F \subseteq \mathbb{N}$, $D\backslash F \in \mathscr{P}(\mathbb{N})$, then

$$\mathscr{E} \subseteq \mathscr{P}(N)$$

Let

$$\mathscr{B} = \{\{n\} : n \in \mathbb{N}\} \;\cup\; \mathscr{E}$$

Since $\{\{n\} : n \in \mathbb{N}\} \subseteq \mathscr{P}(\mathbb{N})$, then $\mathscr{B} \subseteq \mathscr{P}(N)$.

Show that $\mathscr{B}$ is an open base for some topology on ψ.

d) Verify that every subset of ψ is a G_δ set. Hence the ψ-space is first countable.

e) Verify that the ψ-space is a separable space.

Solution to part *a*):

We inductively construct an infinite family of subsets of $\mathbb{N}$, a family such that any pair of its elements has finite intersection.

Let $x_{(0,0)} \in \mathbb{N}$ and $D_0 = \{x_{(0,0)}\}$ and let

$$\mathscr{D}_1 = \{A_{(1,1)}, A_{(1,2)}\}$$

be two infinite subsets of $\mathbb{N}$ such that

$$A_{(1,1)} \cup A_{(1,2)} = \mathbb{N} \text{ and } A_{(1,1)} \cap A_{(1,2)} = \{x_{(0,0}\} = D_0$$

Let

$$D_1 = \{x_{(1,1)}, x_{(1,2)}\} \subseteq \mathscr{D}_1 \backslash D_1$$

where $x_{(1,k)}$ is chosen in $A_{(1,k)} \backslash D_0$, $k = 1, 2$.

Let

$$\mathscr{D}_2 = \{A_{(2,1)}, A_{(2,2)}, A_{(2,3)}, A_{(2,2^2)}\}$$

be a set of infinite subsets such that

$$\begin{array}{lcl} A_{(2,1)} \cup A_{(2,2)} = A_{(1,1)} \backslash D_1 & \text{and} & A_{(2,1)} \cap A_{(2,2)} = \{x_{(1,1)}\} \\ A_{(2,3)} \cup A_{(2,2^2)} = A_{(1,2)} \backslash D_1 & \text{and} & A_{(2,3)} \cap A_{(2,2^2)} = \{x_{(1,2)}\} \end{array}$$

For $k = 1$ to 2^2, choose points $x_{(2,k)} \in A_{(2,k)} \backslash D_1$ and define

$$D_2 = \{x_{(2,1)}, x_{(2,2}, x_{(2,3)}, x_{(2,2^2)}\} \subseteq \cup \mathscr{D}_2 \backslash D_1$$

Note that $D_2 \cap D_1 = \varnothing$.

More generally, at the n^{th} level, let

$$\mathscr{D}_n = \{A_{(n,k)} : k = 1 \text{ to } 2^n\}$$

be a family of 2^n infinite subsets and choose $x_{(n,k)} \in A_{(n,k)} \backslash D_{n-1}$ to form the set

$$D_n = \{x_{(n,k)} : k = 1 \text{ to } 2^n)\} \subseteq \cup \mathscr{D}_n \backslash D_{n-1}$$

We construct from $\mathscr{D}_n$ the $(n+1)^{th}$ level

$$\mathscr{D}_{n+1} = \{A_{(n+1,\, k)} : k = 1 \text{ to } 2^{n+1}\}$$

and choose points $x_{(n+1,k)} \in A_{(n+1,k)} \backslash D_{n-1}$ and define

$$D_{n+1} = \{x_{(n+1,k)} : k = 1, 2, 3, \ldots 2^{n+1}\} \subseteq \cup\{\mathscr{D}_{n+1}\} \backslash D_n$$

as follows:

For each $k = 1$ to 2^n, let $A_{(n+1,2k-1)}$ and $A_{(n+1,2k)}$ be infinite sets such that

$$\begin{aligned} A_{(n+1,2k-1)} \cup A_{(n+1,2k)} &= A_{(n,k)} \backslash D_{n-1} \\ A_{(n+1,2k-1)} \cap A_{(n+1,2k)} &= \{x_{(n,k)}\} \end{aligned}$$

This gives rise to an infinite family of sets

$$\mathscr{D} = \cup\{\mathscr{D}_n : n = 0, 1, 2, 3, \ldots\}$$

where $D_n \subseteq \cup \mathscr{D}_n \backslash D_{n-1}$.

The construction also produces an infinite set of "branches", $\mathscr{B} = \{B_q : q \in J\}$, of infinite subsets of $\mathbb{N}$, such that, for each $q \in J$,

$$B_q = \{x_{(n,q)} : n \in \mathbb{N}\}$$

and

$$x_{(n,q)} \in D_n = \{x_{(n,k)} : k = 1, 2, 3, \ldots, 2^n\}$$

For any pair, s, t, $s \neq t$, the intersection, $B_s \cap B_t$ is finite.

Then $\mathscr{B} = \{B_q : q \in J\}$ is the infinite family of subsets of $\mathbb{N}$ for which every pair has finite intersection.

Solution to part *b*):

We are given that $\overline{\mathscr{F}}$ denotes the collection of all subsets $\mathscr{B}$ of the partially ordered set, $(\mathscr{P}(\mathbb{N}), \subseteq)$, such that every element of $\mathscr{B}$ is infinite and every pair F and G in $\mathscr{B}$, $F \cap G$ is finite. (Note that every element of $\mathscr{B}$ belongs to $\mathscr{P}(\mathbb{N})$, so $\mathscr{B} \in \mathscr{P}(\mathscr{P}(\mathbb{N}).)$ and $\overline{\mathscr{F}} \subseteq \mathscr{P}(\mathscr{P}(\mathbb{N})).$)

We showed in part a) that $\overline{\mathscr{F}}$ is non-empty. We are required to show that $\overline{\mathscr{F}}$ has a maximal element with respect to $\subseteq$.

Let $\overline{\mathscr{C}}$ be a chain of elements in $\overline{\mathscr{F}}$. Let $\mathscr{C}^* = \cup\{\mathscr{B} : \mathscr{B} \in \overline{\mathscr{C}}\}$. We claim that $\mathscr{C}^*$ is an upper bound of $\overline{\mathscr{C}}$ in $\mathscr{F}$. Let $F, G \in \mathscr{C}^*$. Since the elements of $\overline{\mathscr{C}}$ are linearly ordered with respect to $\subseteq$, F and G belong to some $\mathscr{B} \in \mathscr{C}$. Then $F \cap G$ is finite. So $\mathscr{C}^*$ is an upper bound of $\overline{\mathscr{C}}$. So every chain $\overline{\mathscr{C}}$ in $\overline{\mathscr{F}}$ has an upper bound $\mathscr{C}^* \in \overline{\mathscr{F}}$. By Zorn's Lemma $\overline{\mathscr{F}}$ has a maximal element. We will denote it as, $\mathscr{M}$.

Then,

$$\ldots \text{ for any } \mathscr{B} \in \overline{\mathscr{F}},\ \mathscr{B} \subseteq \mathscr{M}$$

To say that $\mathscr{M}$ is "maximal with respect to the given property", means that if E is any infinite subset of $\mathbb{N}$, there exists $D \in \mathscr{M}$ such that $E \cap D$ is infinite (for if not, then we could increase the set $\mathscr{M}$ by one more element).

Solution to part *c*):

It suffices to show that

$$\mathscr{B} = \{\{n\} : n \in \mathbb{N}\} \ \cup\ \{\, \{D\} \cup D \backslash F : D \in \mathscr{M}, F \subseteq_{\text{finite}} D\,\}$$

satisfies the "base property": We must show that $\psi = \cup\{B : B \in \mathscr{B}\}$ and, if $x \in A \cap B$ for some $A, B \in \mathscr{B}$, then there exists $C \in \mathscr{B}$ such that $x \in C \subseteq A \cap B$.

The set $\mathscr{B}$ covers ψ: Let $x \in \psi = \mathbb{N} \cup \mathscr{M}$. If $x \in \mathbb{N}$, then $x \in \{x\} \in \mathscr{B}$. If $x = D \in \mathscr{M}$, then $D \in \{D\} \cup D \backslash F \in \mathscr{E}$. So $\psi \subseteq \cup\{B : B \in \mathscr{B}\}$, hence $\mathscr{B}$ covers ψ.

Case 1. Suppose $A = \{n\}$ and $B = \{n\}$ are both elements of$\mathscr{B}$.

If $x \in \{n\} \cap \{m\}$ then

$$x \ = \ n = m$$

Then there exists $\{x\} \in \mathscr{B}$ such that $x \in \{x\} \subseteq \{n\} \cap \{m\} = \{x\}$.

Case 2. Suppose $A = \{n\}$ and $B = \{D\} \cup D \backslash F$ are both elements of $\mathscr{B}$ and $x \in \{n\} \ \cap\ [\{D\} \cup D \backslash F]$. Then $x = n$ and $x \in D \backslash F$. Then there exists $\{x\} \in \mathscr{B}$ such that $x \in \{n\} = \{x\} \subseteq \{n\} \ \cap\ [\{D\} \cup D \backslash F]$.

Case 3. Suppose $A = \{D_1\} \cup D_1 \backslash F_1$ and $B = \{D_2\} \cup D_2 \backslash F_2$ are both elements of $\mathscr{B}$ and

$$x \in \{D_1\} \cup D_1 \backslash F_1] \ \cap\ [\{D_2\} \cup D_2 \backslash F_2]$$

If $x \in \{D_1\}$ then $x = D_1$. Then $x = D_1 \in \{D_2\}$. Hence $D_2 = D_1$.

See that

$$\begin{aligned}(\{D_1\} \cup D_1\backslash F_1) \;\cap\; (\{D_1\} \cup D_1\backslash F_2) &= \{D_1\} \;\cup\; [(D_1\backslash F_1) \;\cap\; (D_1\backslash F_2)] \\ &= \{D_1\} \;\cup\; (D_1\backslash(F_1 \cup F_2)\end{aligned}$$

Then there exists $\{D_1\} \;\cup\; (D_1\backslash(F_1 \cup F_2) \in \mathscr{B}$ such that

$$x \in \{D_1\} \;\cup\; (D_1\backslash(F_1 \cup F_2) \subseteq (\{D_1\} \cup D_1\backslash F_1) \;\cap\; (\{D_1\} \cup D_1\backslash F_2)$$

Then $\mathscr{B}$ forms an open base for some topology on the set $\psi = \mathbb{N}\cup\mathscr{M}$.[15]

Solution to part *d*):

Clearly every point in $\mathbb{N}$ is a G_δ. Suppose $D \in \mathscr{M}$. If F is a finite subset of $\mathbb{N}$ and $D \in \mathscr{M}$ then $\{D\} = \cap\{\{D\} \cup D\backslash F : F \subseteq_{\text{finite}} D\}$. So ψ is first countable and every point is a G_δ. Let K be a subset of $\mathscr{M}$. Then $T = F \cup \mathbb{N}$ is an open neighborhood of the set F. The set $K = \cap\{T\backslash\{n\} : n \in \mathbb{N}\}$. So K is a G_δ.

Solution to part *e*):

See that every basic open neighborhood of an element D in $\mathscr{M}$ intersects $\mathbb{N}$. Then $\mathbb{N}$ is dense in ψ. So ψ is separable.

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[15] The space (ψ, τ) is referred to as a the ψ-space or the *Mrówka-Isbell space*. Note on interesting facts about ψ in chapters to come. We will see in the example found, later, on page 421 that ψ is Hausdorff locally compact and so is completely regular. From that it will follow that ψ is zero-dimensional. On page 276 we show that the ψ-space is not normal. On page 186 we show that $\mathscr{M}$ is uncountable. The maximality of $\mathscr{M}$ will guarantee that ψ is pseudocompact, as will be shown in the example found on page 399.

Concepts review.

1. Define a neighborhood system of x with respect to τ.
2. Define a neighborhood base of x with respect to the topology τ.
3. Is $\varnothing$ a neighborhood of a point x? Is a neighborhood of x necessarily open?
4. Define a base for a topology τ.
5. Find a base for the usual topology τ on $\mathbb{R}$.
6. Give a characterization of a base of τ in terms of "neighborhoods".
7. What does it mean to say that a subset $\mathscr{P}(S)$ satisfies the "base property"?
8. Describe the Moore plane and the base for its topology.
9. Given an arbitrary subset $\mathscr{S}$ of $\mathscr{P}(S)$ explain how it can be used to construct a topology on S.
10. Define a *subbase for a topology.*
11. Describe a subbase for the usual topology on $\mathbb{R}$.
12. Describe a topology generated by a subbase $\mathscr{S}$ in terms of other topologies on S.
13. Given two topological spaces S and T and a Cartesian product $S \times T$. Find a useful subbase involving projection maps that can be used to generate a topology on $S \times T$.
14. Describe the *lower limit topology* (or Sorgenfrey topology) on $\mathbb{R}$ in terms of its base and subbase.
15. Is $\mathbb{R}$ with the upper limit topology first countable? Is it second countable.
16. What does it mean to say that two subsets of $\mathscr{P}(S)$ are equivalent topologies for the set S.
17. What can we say about the size of the neighborhood bases at the points of $\mathbb{R}$ with respect to the usual topology?
18. What can we say about the size of the base of $\mathbb{R}$ with respect to the usual topology?

19. Define *first countable topological space.*

20. Define *second countable topological space.*

21. Is $\mathbb{R}$ equipped with the usual topology first countable? What about second countable?

22. Describe a topological space which is first countable but not second countable.

23. State a relationship between "second countable" and "separable".

24. Describe the topology on an ordinal space.

25. Metrizable space are necessarily first countable. Describe a neighborhood base.

26. Describe how the set of regular open subsets can be used to generate a topology on a space.

27. What does it mean to say that a property is hereditary?

28. Define a semiregular space.

29. Given a topological space, (S, τ), what is the semiregularization of (S, τ)?

30. Define a zero-dimensional space. As an example provide a subspace of $\mathbb{R}$ which is zero-dimensional.

EXERCISES

1. Show that, if F is a closed subset of the topological space (S, τ) and $x \notin F$, then x has a neighborhood which does not intersect F.

2. Let $\vec{x} = (a, b) \in \mathbb{R}^2$ equipped with the usual (Euclidean) topology. For each $q \in \mathbb{R}$, let $B_{\vec{x}}(q) = \{(x, y) : \max\{|a - x|, |b - y|\} < q\}$. Show that $\mathscr{B}_{\vec{x}} = \{B_{\vec{x}}(q) : q \in \mathbb{R}, q \geq 0\}$ forms a neighborhood base at $\vec{x}$.

3. Suppose τ_1 and τ_2 are two topologies on the set S with respective bases $\mathscr{B}_1$ and $\mathscr{B}_2$. Show that $\tau_1 \subseteq \tau_2$ if and only if whenever $x \in B_1$ there exists $B_2 \in \mathscr{B}_2$ such that $x \in B_2 \subseteq B_1$.

4. Let A and B be two infinite sets each equipped with the cofinite topology. Describe a basis of $A \times B$ equipped with the product topology.

5. Let (S, τ) be a topological space with base $\mathscr{B}$. If $A \subseteq S$, show that $\mathscr{B}_A = \{B \cap A : B \in \mathscr{B}\}$ is a base for the open sets of A.

6. Let (S, τ_S) and (T, τ_T) be two first countable topological spaces. Show that $S \times T$ equipped with the product topology is first countable.

7. Prove that, if (A, τ_1) is a subspace of (B, τ_2) and (B, τ_2) is a subspace of (C, τ_3), then (A, τ_1) is a subspace of (C, τ_3).

6 / Continuity on topological spaces.

Abstract. *In this section we formally define the notion of a continuous function mapping one topological space into another. We discuss various characterizations of these. An important subclass of continuous functions is the one called "homeomorphisms". We will see why these are fundamental in the study of topological spaces.*

6.1 Basic notions and notation associated to functions mapping sets to sets.

We begin by establishing the notation and terminology we will use in our discussion of functions. Suppose $f : S \to T$ is a well-defined function mapping a topological space, S, into another topological space, T. For $f : S \to T$ the expressions, "f is one-to-one and onto" and "f is a bijection between S and T" are simply different ways of conveying the same idea.

The function, $f : S \to T$, induces another function, $f : \mathscr{P}(S) \to \mathscr{P}(T)$, where

$$f[A] = \{y \in T : y = f(x) \text{ for some } x \in A \subseteq S\}$$

Essentially, $f[A]$ is the image of the set A under the function $f : S \to T$. The function $f : S \to T$ also induces the function $f^{\leftarrow} : \mathscr{P}(T) \to \mathscr{P}(S)$ where

$$f^{\leftarrow}[B] = \{x \in S : \text{ where } f(x) \in B\}$$

We can also say that, if $f^{\leftarrow}[B] = D$, then D is the "inverse image" or "pre-image" of B under f, or that the function $f^{\leftarrow}$ "pulls back" the set B onto the set D inside the domain S of f. In the case where $f^{\leftarrow}[\{y\}] = \{x\}$, if there is no risk of confusion, we will simply write $f^{\leftarrow}(y) = x$.[1] In the case where $f : S \to T$ is one-to-one and onto T, then $f^{\leftarrow}$ can itself be seen as a well-defined function, $f^{\leftarrow} : T \to S$, and so we can write "$f^{\leftarrow}(x) = y$ if and only if $f(y) = x$" (without using the square brackets).

In this book, if $f : S \to \mathbb{R}$ is a function and $0 \notin f[S]$, f^{-1} will be interpreted as follows:

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[1] Note that f need not be one-to-one on all of the domain in order for us to speak of $f^{\leftarrow}$ in this way.

$$f^{-1}(x) = \frac{1}{f(x)}$$

In the following theorem statement, we review some basic principles on how functions act on sets. In particular, we are reminded of the following three principles:

1) A function "respects arbitrary unions of sets".
2) If a function is *one-to-one* it will "respect arbitrary intersections". Otherwise, a function "does *not* always respect intersections of sets".
3) An inverse function, $f^{\leftarrow}$, "always respects unions, intersections and complements of sets".

Theorem 6.1 Let $f : A \to B$ be a *function* mapping the set A to the set B. Let $\mathscr{A}$ be *a set* of subsets of A and $\mathscr{B}$ be *a set* of subsets of B. Let $D \subseteq A$ and $E \subseteq B$. Then:

(a) $f\left[\bigcup_{S\in\mathscr{A}} S\right] = \bigcup_{S\in\mathscr{A}} f[S]$

(b) $f\left[\bigcap_{S\in\mathscr{A}} S\right] \subseteq \bigcap_{S\in\mathscr{A}} f[S]$ where equality holds true only if f is one-to-one.

(c) $f[A\backslash D] \subseteq B\backslash f[D]$. Equality holds true only if f is one-to-one and onto B.

(d) $f^{\leftarrow}\left[\bigcup_{S\in\mathscr{B}} S\right] = \bigcup_{S\in\mathscr{B}} f^{\leftarrow}[S]$

(e) $f^{\leftarrow}\left[\bigcap_{S\in\mathscr{B}} S\right] = \bigcap_{S\in\mathscr{B}} f^{\leftarrow}[S]$

f) $f^{\leftarrow}[B\backslash E] = A\backslash f^{\leftarrow}[E]$

Proof:

(a)

$$\begin{aligned} x \in f\left[\bigcup_{S\in\mathscr{A}} S\right] &\Leftrightarrow x = f(y) \text{ for some } y \in \bigcup_{S\in\mathscr{A}} S \\ &\Leftrightarrow x = f(y) \text{ for some } y \text{ in some } S \in \mathscr{A} \\ &\Leftrightarrow x = f(y) \in f[S] \text{ for some } S \in \mathscr{A} \\ &\Leftrightarrow x \in \bigcup_{S\in\mathscr{A}} f[S] \end{aligned}$$

(b) It will be helpful to first prove this statement for the intersection of only two sets U and V. The use of a Venn diagram will also help visualize what is happening.

So we first prove the statement: $f[U \cap V] \subseteq f[U] \cap f[V]$ with equality only if f is one-to-one on $U \cup V$.

Case 1: We consider the case where $U \cap V = \varnothing$.
Then $f\,[U \cap V] = \varnothing \subseteq f[U] \cap f[V]$. So the statement holds true.

Case 2: We now consider the case where $U \cap V \neq \varnothing$.

$$\begin{aligned} x \in f\,[U \cap V] &\Leftrightarrow x = f(y) \text{ for some } y \in U \cap V \\ &\Leftrightarrow x = f(y) \text{ for some } y \text{ contained in both } U \text{ and } V \\ &\Rightarrow x = f(y) \in f[U] \text{ and } f[V] \\ &\Leftrightarrow x \in f[U] \cap f[V] \end{aligned}$$

We now show that, if f is one-to-one on $U \cup V$, then $f[U] \cap f[V] \subseteq f\,[U \cap V]$ and so equality holds true.

- Suppose $x = f(y) \in f[U] \cap f[V]$. Then there exist $u \in U$ and $v \in V$ such that $f(u) = f(v) = f(y)$. Since f is one-to-one, $u = v = y$. This implies $y \in U \cap V$. Hence, $f[U \cap V] = f[U] \cap f[V]$.

The proof of the general statement is left as an exercise.

(c) Proof is left as an exercise.

(d) $$\begin{aligned} x \in f^{\leftarrow}\left[\bigcup_{S \in \mathscr{B}} S\right] &\Leftrightarrow x = f(y) \text{ for some } y \in \bigcup_{S \in \mathscr{B}} S \quad \text{(By definition of } f^{\leftarrow}.) \\ &\Leftrightarrow x = f(y) \text{ for some } y \text{ in some } S \in \mathscr{B} \\ &\Leftrightarrow x \in f^{\leftarrow}[\{y\}] \subseteq f^{\leftarrow}[S] \text{ for some } S \in \mathscr{B} \\ &\Leftrightarrow x \in \bigcup_{S \in \mathscr{B}} f^{\leftarrow}\,[S] \end{aligned}$$

Thus, $f^{\leftarrow}\left(\bigcup_{S \in \mathscr{B}} S\right) = \bigcup_{S \in \mathscr{B}} f^{\leftarrow}\,(S)$.

(e) Proof is left as an exercise.

(f) Proof is left as an exercise.

6.2 Continuous functions on topological spaces.

Given two topological spaces, (S, τ_S) and (T, τ_T), we will discuss various types of functions, $f : S \to T$, which map S into T. The reader is already familiar with those functions called "continuous functions" mapping $\mathbb{R}$ to $\mathbb{R}$. We will generalize this notion of continuity to topological spaces.

Our formal definition of a continuous function mapping a topological space into another is presented below. Those readers who are familiar with the "epsilon-delta" definition of a continuous function (normally presented in any *Introduction to analysis* course) will notice the analytical approach cannot be used in topology since topological spaces, in their most rudimentary form are not equipped with distance functions such as absolute values, norms or metrics.

Definition 6.2 Let $f : S \to T$ be a function mapping (S, τ_S) into (T, τ_T).

(a) We say that

"$f : S \to T$ *is continuous on* S"

if, for any open subset U in T, $f^{\leftarrow}[U]$ is open in S.

(b) If $x \in S$, we will say that

"f is *continuous at the point* x"

if, for any neighborhood, U, of $f(x)$ (inside T) there exists a neighborhood V of x such that $f(x) \in f[V] \subseteq U$.

So a function $f : S \to T$ is continuous if f pulls back open sets in T to open sets in S".

The above definition has two parts to it. The first describes continuity of f on a set, while the second describes continuity of f at a point x. Clearly, we cannot apply the first definition to determine continuity of f at a point. But a set A is simply a collection of points. If a function f can be shown to be continuous at every point x in a set A we would hope that we would obtain continuity on the set as defined in part (a). The next theorem confirms that this is the case.

Theorem 6.3 Let (S, τ_S) and (T, τ_T) be two topological spaces and $f : S \to T$ be a function. Then f is continuous on S if and only if f is continuous at every point of S.

Proof:

> ($\Rightarrow$) Suppose f is continuous on S, $x \in S$ and $y = f(x) \in f[S]$. We are required to show that f is continuous at the point x.
>
> Suppose U is a neighborhood of y. By definition of continuity on a set, $f^{\leftarrow}[\text{int}_T U]$ is an open neighborhood of $x \in f^{\leftarrow}[\{y\}] \subseteq f^{\leftarrow}[\text{int}_T U]$ in S. By definition of neighborhood, there exists an open set, $V \subseteq f^{\leftarrow}[\text{int}_T U]$, such that $x \in V$. Then $y = f(x) \in f[V] \subseteq U$. So we have found the required neighborhood, V, of x. So f is continuous at the point x.
>
> ($\Leftarrow$) Suppose that f is continuous at every point of S. Let U be a non-empty open subset of $f[S] \subseteq T$. We are required to show that $f^{\leftarrow}[U]$ is open in S.
>
> Let $x \in f^{\leftarrow}[U]$. Then $f(x) \in U$. Since f is continuous at x, then there exists a neighborhood V of x such that $f[V] \subseteq U$. Now $x \in \text{int}_T V \subseteq f^{\leftarrow}[f[V]] \subseteq f^{\leftarrow}[U]$. Then $f^{\leftarrow}[U]$ is an open subset of S. So f is continuous on S.

There are other ways of recognizing those functions which are continuous on a set. For example, if $f : S \to T$ satisfies the property,

"$f^{\leftarrow}[F]$ *is closed in* S *whenever* F *is closed in* T"

then, when U is open in T, $S \setminus f^{\leftarrow}[U] = f^{\leftarrow}[T \setminus U]$ is closed in S. Then $f^{\leftarrow}[U]$ is open in S and so f is continuous on S. The reader is left to verify that the converse also holds true.

Other useful characterizations of continuity on a topological space are given below.

Theorem 6.4 Let (S, τ_S) and (T, τ_T) be two topological spaces and $f : S \to T$ be a function.

(a) The function f is continuous on S if and only if f pulls back subbase elements of the topological space T to open sets in S.[2]

---

[2] That is, $f^{\leftarrow}[B]$ is open in S whenever B is a subbase element of T.

(b) The function f is continuous on S if and only if f pulls back open base elements of the topological space T to open sets in S.[3]

(c) The function f is continuous on S if and only if, for any subset U of S,

$$f[\mathrm{cl}_S U] \subseteq \mathrm{cl}_T f[U]$$

Proof: The proofs of parts (a) and (b) are left as an exercise.

c) ($\Rightarrow$) Suppose f is continuous on S. To show that $f[\mathrm{cl}_S U] \subseteq \mathrm{cl}_T f[U]$ it will suffice to show that $x \notin \mathrm{cl}_T f[U]$ implies $x \notin f[\mathrm{cl}_S U]$.

Suppose $x \notin \mathrm{cl}_T f[U]$. Then there musts exist an open V in T such that $x \in V \subseteq T \backslash \mathrm{cl}_T f[U]$. Then $f^{\leftarrow}(x) \subseteq f^{\leftarrow}[V] \subseteq S \backslash U$. By continuity of f, $f^{\leftarrow}[V]$ is open in S and so $\mathrm{cl}_S U \cap f^{\leftarrow}[V] = \varnothing$. Then $f[f^{\leftarrow}(x)] = x \notin f[\mathrm{cl}_S U]$. So $f[\mathrm{cl}_S U] \subseteq \mathrm{cl}_T f[U]$.

The proof of ($\Leftarrow$) is left as an exercise.

Suppose A is a subspace of the topological space, (S, τ), equipped with the subspace topology, τ_A. Suppose $f : S \to T$ is known to be continuous on its domain. Then, when f is restricted to $f|_A$ on A, $f|_A$ preserves the continuity property on A. This is confirmed by the following theorem statement.

Theorem 6.5 Let (S, τ_S), (T, τ_T) and (Z, τ_Z) be topological spaces.

(a) If $f : S \to T$ and $g : T \to Z$ are both continuous on their domains, then $g \circ f : S \to Z$ is continuous on S. (That is, the composition of continuous functions is continuous.)

(b) Suppose $f : S \to T$ is a continuous function on S and $A \subseteq S$. Let $f|_A : A \to T$ denote the restriction of f to the subset A. Then $f|_A$ is continuous on A.

Proof:
(a) The proof of this part is left as an exercise.

---

[3] That is, $f^{\leftarrow}[B]$ is open in S whenever B is a base element of T.

(b) Let U be an open subset of $f[A]$ with respect to the subspace topology $\tau_{f[A]}$ in T. Then there exists $U^* \in \tau_T$ such that $U = U^* \cap f[A]$. Now

$$\begin{array}{rcl} x \in f|_A^{\leftarrow}[U] & \Rightarrow & \{x \in A : f(x) \in U\} \\ & \Rightarrow & x \in \{x \in S : f(x) \in U^*\} \cap A \\ & \Rightarrow & x \in f^{\leftarrow}[U^*] \cap A \end{array}$$

Since f is continuous on S, $f^{\leftarrow}[U^*] \cap A$ is an open subset of A with respect to the subspace topology. So $f|_A^{\leftarrow}[U]$ is open in A.

We have seen that continuous functions are those functions which "pull back" open sets to open sets, or, equivalently, "pull back" closed sets to closed sets. We will encounter at least two other similar types of functions, which are not necessarily continuous.

Definition 6.6 Let $f : S \to T$ be a function mapping the topological space, (S, τ_S), onto the space, (T, τ_T). We say that $f : S \to T$ is *an open function on* S, if, for any open subset U of S, $f[U]$ is open in $f[S]$. We say that $f : S \to T$ is *a closed function on* S, if, for any closed subset F of S, $f[F]$ is closed in $f[S]$.[4]

The reader should be alerted to the fact that an open function need *not* be a continuous function; similarly, a closed function need *not* be continuous. Also, we caution the reader by pointing out that, even if $f : S \to T$ is an open function, it does *not* necessarily follow that $f^{\leftarrow} : T \to S$ is a continuous function on T. (See one of the examples that will follow.) However, for one-to-one functions f, "f is closed if and only if f is open" is true, as we shall now prove.

Suppose $f : S \to T$ is *one-to-one and onto* T. Then f is an open function if and only if f is a closed function. To see this, suppose $f : S \to T$ is a one-to-one and onto open function. We are required

---

[4] Whether the function $f : S \to T$ is open or closed depends on the topology defined on S and T.

to show that f is also a closed function. Let F be a closed subset of S. Then $U = S\backslash F$ is open in S and so $f[U]$ is open in T. Then

$$\begin{aligned} f[U] &= f[S\backslash F] \\ &= f[S]\backslash f[F] \quad \text{(Since } f \text{ is one-to-one.)} \\ &= T\backslash f[F] \end{aligned}$$

Since $T\backslash f[F]$ is open in T, then $f[F]$ is closed in T. We conclude that f is a closed function, as required. Proof of the converse is left to the reader.

Example 1. Suppose $(\mathbb{R}, \tau)$ denotes the real line with the usual topology, τ, and $(\mathbb{R}, \tau_S)$ denotes the real line with the *upper limit topology*, τ_S.

Let $i : (\mathbb{R}, \tau) \rightarrow (\mathbb{R}, \tau_S)$ denote the identity map, $i(x) = x$. Verify that this identity map is open on $(\mathbb{R}, \tau)$ but is not continuous on its domain.

Solution: Since the function $i(x) = x$ maps the open base element, (a, b), of τ to the open set $(a, b) \in \tau_S$ (as we have seen earlier, $\tau \subset \tau_s$), then i maps open sets to open sets hence, i is an open function. But $(a, b\,] \notin \tau$. So $i^{\leftarrow}[(a, b\,]] = (a, b\,] \notin \tau$. So the open identity map i is *not* continuous on $\mathbb{R}$ with respect to τ.

This example illustrates that, if the codomain has more open sets, then the domain, then the identity function will not pass the test of continuity.

Example 2. Let $(\mathbb{R}^2, \tau)$ be equipped with the usual topology. Suppose the open ball center (0, 0) of radius 1,

$$B = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$$

is equipped with the subspace topology, τ_B. We define the function $f : B \rightarrow \mathbb{R}^2$ as follows:

$$f(x, y) = (x, y)$$

If U is an open subset of B, then $f[U] = U \cap B$ an open subset of $\mathbb{R}^2$. So f is an open map. But f is not a closed map. To see this note that B is a closed subset of itself with respect to τ_B. On the other hand, $f[B] = B$ is not closed in the codomain $\mathbb{R}^2$.

If f is one-to-one and onto, we arrive at a different conclusion, as we shall now see.

Theorem 6.7 Let (S, τ_S) and (T, τ_T) be topological spaces. For a one-to-one and onto continuous function $f : S \to T$,

$$f^{\leftarrow} : T \to S \text{ is continuous} \qquad \Leftrightarrow \qquad f : S \to T \text{ is open}$$

Proof: The proof is left as an exercise.

6.3 Homeomorphic topological spaces.

For a continuous function, $f : S \to T$, its inverse function, $f^{\leftarrow} : T \to S$, may, *or may not*, be continuous, even if f is one-to-one. Those one-to-one continuous functions $f : S \to T$ where $f^{\leftarrow}$ is continuous on $f[S] \subseteq T$ are fundamental in the study of topology. They have a special name.

Definition 6.8 Let (S, τ_S) and (T, τ_T) be topological spaces and $f : S \to T$ be a function. If f simultaneously satisfies all three of the following conditions,

1. f is one-to-one on S and onto T
2. f is continuous on S
3. $f^{\leftarrow}$ is continuous on T

then the function, f, is called a *homeomorphism from* S *onto* T. If $f : S \to T$ is a homeomorphism, then S and T are said to be *homeomorphic topological spaces*, or f is said to *map* S *homeomorphically onto* T.

Example 3. Let the open interval, $S = (-\pi/2, \pi/2)$, be equipped with the usual subspace topology. The one-to-one and onto function, $\tan : S \to \mathbb{R}$, is well-known to be continuous on its domain, S. Similarly its inverse (arctan), $\tan^{\leftarrow} : \mathbb{R} \to S$, is continuous on its domain, $\mathbb{R}$. By definition, tan is a homeomorphism. We can then say that $S = (-\pi/2, \pi/2)$ and $\mathbb{R}$ are homeomorphic topological spaces. In fact, as we shall see, any non-empty open interval of $\mathbb{R}$ is homeomorphic to $\mathbb{R}$.

Theorem 6.9 Let (S, τ_S) and (T, τ_T) be topological spaces. Suppose $f : S \to T$ is *one-to-one and onto* S. The following are equivalent:

1. The function f is a homeomorphism.
2. The function f is both continuous and open.
3. The function f is both continuous and closed.

Proof: The proof follows from the statement in Theorem 6.7. It is left as an exercise.

When the function $f : S \to T$ is one-to-one and open but is *not* onto T, the pair S and T cannot be said to be homeomorphic spaces. In this case S is homeomorphic to the proper subspace, $f[S]$, of T. We often express this situation by saying that "f embeds S in T". We define this formally.

Embeddings. Let (S, τ_S) and (T, τ_T) be topological spaces and $f : S \to T$ be a function mapping S to a proper subset of T. If the function $f : S \to f[S] \subset T$ is a homeomorphism, then we say that

"*f embeds S into T*"

or, simply that "f is an *embedding*"[1]. When the specific homeomorphism in question is not explicitly described, we might simply say that

"*T contains a homeomorphic copy of S*"

For example, the function, $f = \frac{2}{\pi}\arctan$, embeds $\mathbb{R}$ into $[-1, 1]$ since $f[\mathbb{R}] = (-1, 1) \subset [-1, 1]$.

This homeomorphic copy is, of course, the image of the hypothesized homeomorphism.

Also, note that, if $f : S \to T$ embeds S into T, then $f^{\leftarrow} : f[S] \to S$ is also a homeomorphism. In general, we can say that

---

[1] Some authors use the spelling "imbeds" rather than "embeds", which is also correct. One should be careful on how one uses the word "embed". For example, suppose $f : S \to T$ embeds S into T and $g : T \to S$ embeds T into S, can we conclude that S and T are homeomorphic spaces? Consider the functions $f(x) = x/2$ and the function $g(x) = x$ and the sets $S = (0, 1)$ and $T = (0, 1/2) \cup (1/2, 1)$. Verify that f embeds S into T and that g embeds T into S. Are S and T homeomorphic?

> ...“*the function, $f : S \to T$, is a homeomorphism if and only if there is a continuous map $g : T \to S$ such that $g \circ f$ and $f \circ g$ are the identity map on S and T, respectively.*

We have seen that, if $f : S \to T$ is a homeomorphism between the two topological spaces (S, τ_S) and (T, τ_T), the one-to-one function, f, maps each open base element in τ_S to a unique set in τ_T, and vice-versa. Since f is one-to-one it respects both arbitrary unions and arbitrary intersections; then every element of τ_S will be paired, under f, to exactly one element in τ_T. This suggests that some properties in S which involve open sets will be “mirrored” inside T. We will refer to such properties as “topological properties”. We formally define this notion.

Definition 6.10 Suppose P is a property which is satisfied on a topological space S. If P is satisfied in any homeomorphic copy of S, then P is called a *topological property* or a *topological invariant*.

If $f : S \to T$ is a homeomorphism and P is known to be a topological property, then, by definition,

P is satisfied in $f[S]$ if and only if P is satisfied in S

Example 4. Recall that in a metric space, (S, ρ), a sequence, $\{x_n : n \in \mathbb{N}\}$, is referred to as being a Cauchy sequence if and only if, for every $\varepsilon > 0$, there exists N such that whenever $n, m > N$, then $\rho(x_n, x_m) < \varepsilon$. Being a “Cauchy sequence” in a metric space is a property which does not translate easily to a topological space. To see this consider the continuous function $g(x) = 1/x$ on $\mathbb{R}\backslash\{0\}$. The function g maps $\mathbb{R}\backslash\{0\}$ homeomorphically onto itself. The sequence $T = \{1, 1/2, 1/3, \ldots,\}$ is a Cauchy sequence in the domain of g. But $g[T] = \{1, 2, 3, \ldots,\}$ is not Cauchy in the image of g in $\mathbb{R}\backslash\{0\}$. [5]

Example 5. Consider the following property, P, on a space S: “Every real-valued continuous function on S assumes its maximum value on S” Verify that P is a topological property.

Solution: The negation, $\neg P$, of the property P is: “There is a continuous real-valued function on S which does not attain its maximum

---

[5] We will eventually see that the notion of uniform continuity of a function on a subset of a metric space also does not translate automatically to a topological space.

value in S". We will show that $\neg P$ is a topological property. The desired result will follow.

Suppose S is a space which satisfies $\neg P$. Then there exists a continuous function,

$$f : S \to \mathbb{R}$$

which does not attain its maximum value in S.

Let $h : S \to T$ be a homeomorphism mapping S onto T. We claim that T also satisfies $\neg P$.

Proof of claim: Consider the continuous function, $f \circ h^{\leftarrow} : T \to \mathbb{R}$. If $b \in T$, then $h^{\leftarrow}(b) \in S$. Then, by hypothesis, there must exist $a \in S$ such that $f(a) > (f \circ h^{\leftarrow})(b)$. Note that $(f \circ h^{\leftarrow})(h(a)) = f(a) > (f \circ h^{\leftarrow})(b)$. Thus $f \circ h^{\leftarrow}$ does not assume a maximum value on T. Then T also satisfies $\neg P$, as claimed.

We conclude that S satisfies $\neg P$ if and only if T satisfies $\neg P$. Then S satisfies P if and only if T satisfies P. So P is a topological property.

It is interesting to note that the closed interval $[0, 1]$ cannot be homeomorphic to $\mathbb{R}$ since elementary calculus shows that $[0, 1]$ satisfies property P and $\mathbb{R}$ does not.

6.4 Continuity and countability properties.

We have previously defined the three countability properties,

separable
first countable
second countable

We would like to verify whether or not a continuous function will always carry over each of these properties from its domain into its codomain.

Recall that a topological space, (S, τ), is *separable* if and only if S contains a countable dense subset. We expect that "separable" is a topological property. This fact is confirmed by the following result which shows that separability is carried over by continuous functions. Hence, if S is separable, every topological space which is homeomorphic to S is also separable.

Theorem 6.11 Suppose (S, τ_S) is a separable topological space. Then any continuous image of S is also separable.

Proof: Suppose (S, τ_S) is separable and $f : S \to T$ is a continuous function mapping S onto the topological space (T, τ_T).

We are required to show that the image, T, of S under f is also separable. That is, that T has a countable dense subset. Given that S is separable, then it contains a countable dense subset, say D.

We claim that $f[D]$ is a countable dense subset of $T = f[S]$.
Proof of claim: Since the cardinality of the image of a function is less than or equal to the cardinality of its domain, then $f[D]$ is a countable subset of T. It now suffices to show that $f[D]$ is dense in T. Suppose U is a non-empty open subset of T. Continuity of f guarantees that $f^{\leftarrow}[U]$ is open and so there must exist $x \in f^{\leftarrow}[U] \cap D$. Then

$$\begin{aligned} f(x) &\in f[f^{\leftarrow}[U] \cap D] \\ &\subseteq f[f^{\leftarrow}[U]] \cap f[D] \\ &= U \cap f[D] \end{aligned}$$

Hence, every open subset U of T intersects $f[D]$ in a non-empty set. We conclude that $f[D]$ is dense in T. So $T = f[S]$ is a separable space.

So homeomorphisms carry over the separable property.

A simple continuous function is (by itself) not quite strong enough to carry over the second countable property nor the first countable property from its domain to its codomain. We suspect that homeomorphisms can certainly do so.

But, as we shall see, we don't need the full power of a homeomorphism for this. The following theorem shows that a continuous *open* function will suffice. If so, since homeomorphisms are both continuous and open, then "second countable" and "first countable" are topological properties.

Theorem 6.12 Let $f : S \to T$ be a continuous *open* function mapping S onto a space T. If the space S is second countable, then T is second countable. If the space S is first countable, then T is first countable.

Proof: Let (S, τ_S) and (T, τ_T) be topological spaces and $f : S \to T$ be a *continuous open function* mapping S onto T.

If S is second countable, then S has a countable open base, $\mathscr{B}$. Consider the family, $\mathscr{B}_T = \{f[B] : B \in \mathscr{B}\}$. By hypothesis, $\mathscr{B}_T$ is a set of countably many open subsets of T. We claim that the countable set, $\mathscr{B}_T$, is a base for open sets in T.

Proof of claim: Let U be a non-empty open subset of T and let $y \in U$. Let $x \in f^{\leftarrow}(y) \in f^{\leftarrow}[U]$. Since f is continuous, $f^{\leftarrow}[U]$ is open in S, so there exist an open $V \in \mathscr{B}$ such that $x \in V \subseteq f^{\leftarrow}[U]$. Since f is declared to be open, $f(x) = y \in f[V] \in \mathscr{B}_T$, an open subset of U. We can then conclude that U is the union of elements from $\mathscr{B}_T$. So $\mathscr{B}_T$ forms a base for T, so T is second countable.

The proof that the "first countable" property is carried over by continuous open functions is left as an exercise.

6.5 The weak topology induced by a family of functions.

Suppose (T, τ_T) is a space and we are given a function, $f : S \to T$, mapping the set S onto the topological space, T (where S is not yet topologized). With this premise alone, we would normally not discuss the continuity of f since "continuity" is defined in terms of a topology on both the function's domain and codomain. But we could first hypothesize "continuity of f" and then force on S enough open sets so that this family of open sets would support the continuity of f.

Is this hard to do? Not really. We need only equip the domain, S of f, with the discrete topology, τ_d. Then, no matter how f is defined, since every subset of S is open, then f is, by definition, continuous on S. But most readers would not find this answer entirely satisfactory − nor an interesting problem − since the proposed topology on S doesn't depend on the function, f, at all. We can tighten up our question a bit so that the topology on S will relate specifically to the function f in question.

Let $f : S \to T$ be some function mapping the set S into the topological space T. We will topologize S so that, not only is the function f guaranteed to be continuous on S, but that the chosen topology is the "smallest possible topology that will guarantee the continuity of f". To do this, we will define

$$\mathscr{S}_f = \{f^{\leftarrow}[U] : U \in \tau_T\}$$

a subset of $\mathscr{P}(S)$, declaring it to be *a subbase for a topology on S*. By taking all finite intersections of elements from $\mathscr{S}_f$ we will generate the subset, $\mathscr{B}_f$, of $\mathscr{P}(S)$ which satisfies the "base property". By taking all unions of elements from $\mathscr{B}_f$ we will generate the smallest topology, τ_f, with subbase $\mathscr{S}_f$. The family τ_f is the weakest topology possible that will guarantee the continuity property on f. We will refer to τ_f as being the

weak topology on S induced by f

Eliminating just one element from this topology would be sufficient to prevent f from being continuous on S. Also, the topology we obtain is directly related to the function f. If we consider a different function we may obtain a different topology.

If we want the two functions $f : S \to T$ and $g : S \to T$ to be continuous on S, we would require more open sets on S. That is, the required subbase would have to be,

$$\mathscr{S}_{\{f,g\}} = \{f^{\leftarrow}[U] : U \in \tau_T\} \cup \{g^{\leftarrow}[U] : U \in \tau_T\}$$

In this case the topology generated by the subbase $\mathscr{S}_{\{f,g\}}$ would be called the *weak topology induced by* $\{f,\ g\}$.

We can generalize this even more. Rather than restrict ourselves to only two functions each with the same domain S, we will consider the more general "*weak topology on S induced by a family of functions* $\{f_\alpha : S \to T_\alpha\}_{\alpha \in I}$", formally defined below.

Definition 6.13 Let S be a non-empty set and $\{(T_\alpha, \tau_\alpha) : \alpha \in \Gamma\}$ denote a family of topological spaces. For each $\alpha \in \Gamma$, suppose

$$f_\alpha : S \to T_\alpha$$

is a function mapping S onto T_α. Let

$$\mathscr{S} = \{f_\alpha^{\leftarrow}[U_\alpha] : U_\alpha \in \tau_\alpha\}_{\alpha \in \Gamma}$$

Then $\mathscr{S}$ can be declared to be the subbase of a topology, $\tau_{\mathscr{S}}$, on S. The topology $\tau_{\mathscr{S}}$ is called the

"*weak topology induced by the family of functions* $\{f_\alpha : \alpha \in \Gamma\}$ on S"

The reader is encouraged to keep this definition in mind since we will refer to the weak topology induced by a set of functions when we define a suitable topology on the Cartesian product of a family of topological spaces.

6.6 Topic: Continuity on a dense subset of a topological space.

We examine particular properties of a continuous function on a dense subset of (S, τ).

Note that, if A is dense in B and $A \subseteq D \subseteq B$, then D must be dense in B. The reader is left to verify this. It is also easy to verify that, if D is dense in S, then $S \backslash D$ cannot contain any non-empty open subsets of S.

Before we state the next theorem we define the following commonly used expression: Given two functions, $f : S \to T$ and $g : S \to T$ we say that

"f and g agree on A"

if $f(x) = g(x)$ for all $x \in A \subseteq S$.

Theorem 6.14 Suppose (S, τ) is a topological space and (T, τ_ρ) is a metrizable topological space induced by the metric ρ. Suppose $f : S \to T$ and $g : S \to T$ are two continuous functions which agree on some dense subset D of (S, τ). Then f and g must agree on all of S.[6]

Proof: Let (T, ρ) be the metric space which is equivalent to (T, τ_ρ). Suppose the continuous functions $f : S \to T$ and $g : S \to T$ agree on a dense subset D of S. We are required to show that f and g agree on S.

Suppose $a \in S \backslash D$ such that $f(a) \neq g(a)$ in T. We will show this leads to a contradiction. Then there exist in T, *disjoint* basic open balls, $B_\varepsilon(f(a))$ and $B_\varepsilon(g(a))$, of S of radius

$$\varepsilon = \frac{\rho(\, f(a),\, g(a)\,)}{3}$$

with center $f(a)$ and $g(a)$, respectively. Since both f and g are continuous on S, both $f^{\leftarrow}[B_\varepsilon(f(a))]$ and $g^{\leftarrow}[B_\varepsilon(g(a))]$ are open in S each containing at least the point a. So, the open subset

$$a \in f^{\leftarrow}[B_\varepsilon(f(a))] \cap g^{\leftarrow}[B_\varepsilon(g(a))] \neq \varnothing$$

---

[6] Once we have introduced the concept of "Hausdorff" this statement generalizes from "metrizable topological spaces" to "Hausdorff topological spaces".

in S. Since D is dense in S, then $f^{\leftarrow}[B_\varepsilon(f(a))] \cap g^{\leftarrow}[B_\varepsilon(g(a))] \cap D \neq \varnothing$. If $x \in f^{\leftarrow}[B_\varepsilon(f(a))] \cap g^{\leftarrow}[B_\varepsilon(g(a))] \cap D$, then $f(x) \neq g(x)$ contradicting the fact that f and g agree on D. So there can be no point $a \in S \backslash D$ such that $f(a) \neq g(a)$ in T.

So f and g must agree on S.

Example 6. Suppose S is a topological space whose points are closed in S. Suppose S is such that, for any closed subset K, any point $x \notin K$ is guaranteed to have a closed neighborhood F which misses K. Let D be a dense subset of S. Suppose $f : S \to T$ is a continuous function mapping S into a topological space T such that $f|_D$ embeds D into T, but is not a homeomorphism on all of S. Show that $f[S\backslash D] \subseteq T\backslash f[D]$.

Solution: Suppose $f[S\backslash D] \not\subseteq T\backslash f[D]$.

That is, suppose there is some $y \in S\backslash D$ such that $f(y) \in f[D]$. Then there is $x \in D$ such that $f(y) = f(x)$. We will show that this leads to a contradiction.

By hypothesis, x has a closed neighborhood, F, which misses the closed subset $\{y\}$. Since D is dense in S and f is a homeomorphism on D, $f[F \cap D]$ is a neighborhood of $f(x)$ in $f[D]$.

Since F is closed in S, by hypothesis, there is an open neighborhood, U, of y which misses F. Since D is dense in S, $U \cap D \neq \varnothing$ and since f is a homeomorphism on D,

$$[\, f[U] \cap f[D]\,] \cap [\, f[F] \cap f[D]\,] = \varnothing$$

Then, no matter the choice of the neighborhood U of y, $f[U] \not\subset f[F] \cap f[D]$. This contradicts continuity of f at y. The source of the contradiction is our assumption that $f(y) \in f[D]$. Then, if $y \in S\backslash D$, $f(y) \notin f[D]$. So $f[S\backslash D] \subseteq T\backslash f[D]$.

6.7 Topic: On algebras of continuous real-valued functions.

We now take a quick glance at the algebra[7], $(C(S), +, \cdot, \text{scalar mult})$, of those continuous functions on a topological space, S, with range in $\mathbb{R}$. If S is a topological space, we denote the set of all continuous functions, $f : S \to \mathbb{R}$, mapping S into $\mathbb{R}$ by $C(S)$. The set $C(S)$ is

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[7] An *algebra* is a set with $+$ and $\cdot$ and scalar multiplication by elements of a field.

normally considered equipped with the algebraic operations $+$, $\cdot$ and scalar multiplication defined as

$$\begin{aligned}(f+g)(x) &= f(x)+g(x)\\ (f\cdot g)(x) &= f(x)g(x)\\ (\alpha f)(x) &= \alpha f(x)\end{aligned}$$

We will assume that the reader is able to write proofs showing that $C(S)$ is closed under sums, multiplication and real scalar multiplication and that $|f|$ defined as

$$|f|(x) = |f(x)|$$

also belongs to $C(S)$ whenever f is in $C(S)$. We will also refer to $(C(S), +, \cdot)$ as a *ring of continuous functions*. The elements of $C(S)$ can be *partially ordered* with "$\leq$"[8] where

$$f \leq g \text{ if and only if } f(x) \leq g(x)$$

for all $x \in S$. We also define the operations $\vee$ and $\wedge$ on $C(S)$ as,

$$\begin{aligned}(f\vee g)(x) &= \max\{f(x), g(x)\} \text{ for } x \in S\\ (f\wedge g)(x) &= \min\{f(x), g(x)\} \text{ for } x \in S\end{aligned}$$

Verification of the following formulas is left to the reader.

$$\begin{aligned}f\vee g &= \frac{f+g+|f-g|}{2}\\ f\wedge g &= \frac{f+g-|f-g|}{2}\end{aligned}$$

From these we can conclude that both $f \vee g$ and $f \wedge g$ are continuous whenever f and g are continuous.

6.8 Topic: The Pasting lemma and a generalization.

The continuity of a function, f, on a given space can sometimes be more easily confirmed by determining the continuity of f on some of its subspaces. The "pasting lemma" is a statement which says that two continuous functions can be "pasted together" to create some other continuous function. The lemma is implicit in the use of piecewise defined functions.

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[8] A partially ordered set is a set P on which is defined a binary relation "$\leq$" which is reflexive, antisymmetric and transitive.

Lemma 6.15 Let A and B, be closed subsets of a topological space S such that $S = A \cup B$. Then $f : S \to T$ is continuous if and only if $f|_A$ and $f|_B$ are both continuous.

Proof: Let $f : S \to T$ be a function and A and B be both closed in S.

> ($\Rightarrow$) Suppose f is continuous on S. Then, by Theorem 6.5 both $f|_A$ and $f|_B$ are continuous on A and B, respectively.
>
> ($\Leftarrow$) Suppose both $f|_A$ and $f|_B$ are continuous on the closed subsets A and B, respectively. Let F be a closed subset of T. Then both $f|_A^{\leftarrow}[F]$ and $f|_B^{\leftarrow}[F]$ are closed since each is the pre-image of f when restricted to A and B, respectively. Then their union, is also closed, being a finite union of closed sets. So $f^{\leftarrow}[F] = f|_A^{\leftarrow}[F] \cup f|_B^{\leftarrow}[F]$, a closed subset of S. So f is continuous on S.

> We generalize the Pasting lemma to the case where f acts on an infinite collection of closed sets. The statement in the following theorem involves a family (possibly infinite) of subsets referred to as being a "locally finite collection".

Definition 6.16 A *locally finite* collection, $\mathscr{C}$, of subsets of a topological space, S, is one for which every point in S has at least one neighborhood which meets only finitely many elements of $\mathscr{C}$.

> So a locally finite family $\mathscr{C}$ of subsets of S can be visualized as being one whose elements do not "cluster" about any of the points of S. A locally finite family of subsets is one which satisfies an interesting property, presented in the following lemma. Recall from the example on page 60, that even if $\cup\{\mathrm{cl}_S A_i : i \in I\} \subseteq \mathrm{cl}_S[\cup\{A_i : i \in I\}]$ is always true, it may occur that $\cup\{\mathrm{cl}_S A_i : i \in I\} \neq \mathrm{cl}_S[\cup\{A_i : i \in I\}]$.

Lemma 6.17 Suppose (S, τ) is a topological space. Let $\mathscr{U} = \{U_i : i \in I\}$ be a locally finite collection of sets in S. Then $\mathscr{U}^* = \{\mathrm{cl}_S U_i : i \in I\}$ is also locally finite.

Furthermore,

$$\mathrm{cl}_S[\cup\{U_i : i \in I\}] = \cup\{\mathrm{cl}_S U_i : i \in I\}$$

Proof: Let $\mathscr{U} = \{U_i : i \in I\}$ be a locally finite collection of sets in S.

Then, for $x \in S$, there is an open neighborhood B such that $B \cap U \neq \varnothing$ for finitely many members of $\mathscr{U}$ and $B \cap U = \varnothing$ for all others. For these "other", U's, since B is open in S, $B \cap \mathrm{cl}_S U = \varnothing$. In the case where $B \cap U \neq \varnothing$ then $B \cap \mathrm{cl}_S U \neq \varnothing$ for finitely many U's. The existence of this open neighborhood B of x is all that is required to prove the local finiteness property of $\{\mathrm{cl}_S U_i : i \in I\}$.

We now prove the second part. Let

$$M = \cup\{U_i : i \in I\}$$

We are required to prove $\mathrm{cl}_S M = \cup\{\mathrm{cl}_S U_i : i \in I\}$.

Clearly, $\cup\{\mathrm{cl}_S U_i : i \in I\} \subseteq \mathrm{cl}_S M$. So we need only prove $\mathrm{cl}_S M \subseteq \cup\{\mathrm{cl}_S U_i : i \in I\}$

Let $p \in \mathrm{cl}_S M$. We claim that $p \in \cup\{\mathrm{cl}_S U_i : i \in I\}$. There exists an open neighborhood B of p such that $B \cap \mathrm{cl}_S U_i \neq \varnothing$ for finitely many elements, $\mathscr{F}$, in $\mathscr{U}^*$.

See that, for $p \in \mathrm{cl}_S M$,

$$\begin{aligned} p \in \mathrm{cl}_S M &= \mathrm{cl}_S \left[\cup\{U_i \in \mathscr{U} : \mathrm{cl}_S U_i \notin \mathscr{F}\} \bigcup \cup\{U_i \in \mathscr{F}\}\right] \\ &= \mathrm{cl}_S \left[\cup\{U_i \in \mathscr{U}^* : \mathrm{cl}_S U_i \notin \mathscr{F}\}\right] \bigcup \mathrm{cl}_S \left[\cup\{U_i \in \mathscr{F}\}\right] \\ &= \mathrm{cl}_S \left[\cup\{U_i \in \mathscr{U}^* : \mathrm{cl}_S U_i \notin \mathscr{F}\}\right] \bigcup \cup\{\mathrm{cl}_S U_i \in \mathscr{F}\} \quad \text{(}\mathrm{cl}_S U_i \text{ is closed and} \mathscr{F} \text{ is finite)} \end{aligned}$$

Now p cannot belong to $\mathrm{cl}_S \left[\cup\{U_i \in \mathscr{U}^* : \mathrm{cl}_S U_i \notin \mathscr{F}\}\right]$ since $B \cap \mathrm{cl}_S U_i = \varnothing$ for all $\mathrm{cl}_S U_i \notin \mathscr{F}$. Then $p \in \cup\{\mathrm{cl}_S U_i \in \mathscr{F}\} \subseteq \cup\{\mathrm{cl}_S U_i : i \in I\}$, as claimed.

So $\mathrm{cl}_S M = \cup\{\mathrm{cl}_S U_i : i \in I\}$, as required.

It follows from the lemma that, if $\mathscr{F} = \{F : F \in \mathscr{F}\}$ is any locally finite collection of closed sets, then

$$\mathrm{cl}_S[\cup\{F : F \in \mathscr{F}\} = \cup\{F : F \in \mathscr{F}\}$$

Theorem 6.18 Suppose (S, τ) is a topological space. Let $\mathscr{F}$ be a locally finite family of closed subsets which covers all of S (i.e., $S = \cup\mathscr{F}$). Let $f : S \to T$ be a function mapping S into a space T. Then f is continuous on S if and only if $f|_F : F \to T$ is continuous on each $F \in \mathscr{F}$.

Proof: By Theorem 6.5, (b), we need only prove ($\Leftarrow$).

($\Leftarrow$) Suppose $\mathscr{F}$ is a locally finite collection of closed subsets of S which covers all of S. Also, suppose $f : S \to T$ is a function mapping S into a space T such that $f|_F : F \to T$ is continuous on each $F \in \mathscr{F}$. We are required to show that f is continuous on all of S. To prove this, it suffices to show that, if A is closed in T, then $f^{\leftarrow}[A]$ is closed in S.

Let A be a closed subset of T. Then,

$$\begin{aligned} f^{\leftarrow}[A] &= \cup\{F \cap f^{\leftarrow}[A] : F \in \mathscr{F}\} \\ &= \cup\{f|_F^{\leftarrow}[A] : F \in \mathscr{F}\} \end{aligned}$$

Since $f|_F$ is continuous for each $F \in \mathscr{F}$, then $f|_F^{\leftarrow}[A]$ is a closed subset of F. Then there exists a closed subset G of S such that $f|_F^{\leftarrow}[A] = G \cap F$, a closed subset of S.

Consider the collection

$$\mathscr{C} = \{f|_F^{\leftarrow}[A] : F \in \mathscr{F}\}$$

of closed subsets of S. Note that $f_F^{\leftarrow}[A] = \cup\mathscr{C}$. We will show that this set is closed in S.

Claim: We claim that $\mathscr{C}$ is locally finite.

Proof: Let $p \in S$. Then there exists an open neighborhood, V, of p and a finite subcollection, $\mathscr{G}$, of $\mathscr{F}$ such that $V \cap F \neq \varnothing$ if $F \in \mathscr{G}$ and $V \cap F = \varnothing$ for $F \in \mathscr{F}\backslash\mathscr{G}$.

Note that for $\mathscr{F}\backslash\mathscr{G}$,

$$\begin{aligned} V \cap f|_F^{\leftarrow}[A] &= V \cap (F \cap f^{\leftarrow}[A]) \\ &= (V \cap F) \cap f^{\leftarrow}[A] \\ &= \varnothing \cap f^{\leftarrow}[A] \\ &= \varnothing \end{aligned}$$

Then at most finitely many elements of $\mathscr{C}$ intersect the neighborhood V of p. So $\mathscr{C}$ is locally finite collection of closed subsets, as claimed.

Then $f^{\leftarrow}[A] = \cup\mathscr{C}$ is closed (see paragraph preceding the theorem).

Then $f : S \to T$ is continuous on S.

Note that, in the above theorem, if the members of the collection $\mathscr{F}$ are all *open subsets* (rather than all closed as hypothesized in the theorem) the family $\mathscr{F}$ need not be locally finite for the statement to hold true. That is...

> *Let S and T be topological spaces and $\{O_i : i \in I\}$ be a collection of open subsets of S which covers all of S. Let $f : S \to T$ be a function. Then, $f : S \to T$ is a continuous function on S if and only if the restriction, $f|_{O_i}$, of f to O_i is continuous for each $i \in I$.*

It is left as an easy exercise for the reader to verify that this holds true.

Corollary 6.19 Suppose (S, τ) is a topological space. Let $\mathscr{F} = \{F_i : i \in F\}$ be a *finite* family of closed subsets which covers all of S. Let $f : S \to T$ be a function mapping S into a space T. Then f is continuous on S if and only if $f|_{F_i} : F_i \to T$ is continuous on each $F_i \in \mathscr{F}$.

Proof: See that, if $\mathscr{F} = \{F_i : i \in F\}$ be a *finite* family of closed subsets which covers all of S, then $\mathscr{F}$ is a locally finite cover of closed sets on S. The result follows immediately from the theorem.

Concepts review.

1. Given the function $f : S \to T$ and $A \subseteq S$, define $f[A]$.
2. Given the function $f : S \to T$ and $B \subseteq S$, define $f^{\leftarrow}[B]$.
3. State the formal topological definition of "f is continuous function on the set A".
4. State the formal topological definition of "f is continuous function at the point x".
5. Give a formal theorem statement which links the above two definitions of continuity.
6. A continuous function $f : S \to T$ pulls back open subbase elements in $\mathscr{P}(T)$ to what kind of set in S?
7. A continuous function $f : S \to T$ pulls back open base elements in $\mathscr{P}(T)$ to what kind of set in S?
8. If $f : S \Rightarrow T$ is continuous on S and $A \subseteq S$, show that $f|_A$ is continuous on A.

9. Is it correct to say "$f : S \to T$ is continuous if and only if f pulls back closed sets to closed sets"?

10. What does it mean to say $f : S \to T$ is an open functions?

11. What does it mean to say $f : S \to T$ is a closed functions?

12. Is it okay to say that "open functions are always closed"? What about "continuous functions"?

13. What does it mean to say that $f : S \to T$ is a homeomorphism?

14. What does it mean to say that S and T are homeomorphic spaces?

15. Give two characterizations of homeomorphic functions.

16. What is a topological property?

17. Is "first countable" a topological property? What about "second countable"?

18. Do continuous functions necessarily carry over the second countable property to its codomain? What about the first countable property?

19. What does it mean to say that A is a dense subset of B?

20. What can we say about two continuous functions which agree on a dense subset of a metrizable topological space?

21. What does it mean to say that a topological space is separable?

22. What can we say about continuous images of separable spaces?

23. What can we say about second countable spaces in reference to the "separable property"?

24. Provide examples of hereditary and non-hereditary topological properties.

25. Define the weak topology induced by a family of functions $\{f_\alpha : S \to T_\alpha\}_{\alpha \in I}$.

26. Provide two characterizations of "$f : S \to T$ is continuous on S" involving families of subsets which cover S.

EXERCISES

1. Suppose $S = \{a, b\}$ is a set with topology $\tau_S = \{\varnothing, \{a\}, S\}$. Let $T = \{a, b\}$ where T is equipped with the discrete topology τ_d. Let $i : S \to T$ denote the identity map. What can we say about the continuity or non-continuity of the functions $i : S \to T$ and $i^{\leftarrow} : T \to S$?

2. Let $f : S \to T$ be a continuous function mapping (S, τ_S) onto (T, τ_T). If U is a G_δ in T, is $f^{\leftarrow}[U]$ necessarily a G_δ in S? Show that f pulls back F_σ's in T to F_σ's in S.

3. Let $f : (S, \tau_S) \to (T, \tau_T)$ be a continuous function mapping S onto T. Show that f is continuous on S if and only if $f[\mathrm{cl}_S[U]] \subseteq \mathrm{cl}_T f[U]$, for any $U \in \mathscr{P}(S)$.

4. Recall that *infinite countable sets* are those sets which can be mapped one-to-one and onto the natural numbers $\mathbb{N}$. Suppose X and Y are both countable dense subsets of $(\mathbb{R}, \tau)$ where τ is the usual topology. Show that X and Y must be homeomorphic subspaces of $\mathbb{R}$.

5. Consider the topological spaces $S = (\mathbb{R}^2, \tau_1)$ and $T = (\mathbb{R}, \tau)$ where τ_1 and τ represent the usual topology. (The open base of τ_1 are the open balls, $B_\varepsilon(x, y)$, with center (x, y) and radius ε). Consider the function $f : S \to T$ defined as $f(x, y) = x$.
    (a) Show that f is an open function.
    (b) Show that f is not a closed function. (Hint: See that $F = \{(x, y) : xy = 1\}$ is a closed subset of S. Consider the image of F under f.)

6) Prove: The function $f : S \to T$ is a closed function if and only if whenever F is a closed subset of S, then $\{t \in T : f^{\leftarrow}[\{t\}] \cap F \text{ is non-empty}\}$ is a closed subset of T.

7) Suppose $f : S \to T$ is a closed function. Show that whenever V is an open subset of S and $f^{\leftarrow}[\{x\}]$ is a subset of V, then $x \in \mathrm{int}_T(f[\mathrm{cl}_S V])$.

8) Suppose $f : S \to T$ is a closed function and F is a closed subset of S. Show that the restriction, $f|_F : F \to f[F]$, is also a closed function.

9) Suppose U and V are subsets of S such that $U \cup V = S$ and $x \in U \cap V$. Suppose $f : S \to T$ is a function such that both $f|_U$ and $f|_V$ are continuous on U and V, respectively. Show that f is continuous at x.

10) Suppose that $f : S \to T$ is a one-to-one and onto function. Show that f is a homeomorphism if and only if, for any $U \in \mathscr{P}(S)$, $f[\mathrm{cl}_S U] = \mathrm{cl}_T f[U]$.

7 / Product spaces.

Abstract. *In this section we will review some fundamental facts about those sets that are "Cartesian products of sets". We will then consider two topologies on a Cartesian product of topological spaces and study some of their most fundamental properties. In the last half of the chapter, we will look at some applications where product spaces play an important role. In particular, we prove the existence of a continuous function which maps the closed interval* $[0,1]$ *onto the product space,* $[0,1]\times[0,1]\times[0,1]$*, a cube in a three dimensional space.*

7.1 Fundamentals of Cartesian products.

In topology, Cartesian products are a rich and important source of examples of various known topological properties. Given a family of known topological spaces, we can construct a new larger topological space with it's own particular properties. For this reason, it is crucial to understand them well. The best way to do this is to practice using them in various contexts.

Most students are exposed to the notion of a Cartesian product with two or three factors, in some form or other, at the high school level. For example,

$$A\times B\times C=\{(a,b,c):a\in A,b\in B,c\in C\}$$

When discussing *infinite* Cartesian products of sets we must proceed cautiously, while deciding which topology we will adopt to best suit our purposes. We thought it would be best if we begin by presenting a formal definition of products of sets. Those readers already well familiar with these concepts can skim through this section and go directly to section 7.2.

Definition 7.1 Let $\{S_\alpha:\alpha\in I\}$ be an indexed family of sets. The Cartesian product of these sets, denoted by $\prod_\alpha S_\alpha$, or in more detail as,

$$\textstyle\prod_{\alpha\in I}S_\alpha=\{\ f\mid f:I\to\cup_{\alpha\in I}S_\alpha\}$$

is the set of all functions f mapping the index set, I, into the union, $\cup_{\alpha\in I}S_\alpha$, such that, for $\beta\in I$, $f(\beta)\in S_\beta$.[1] So, if u is an element of $\prod_\alpha S_\alpha$ and

---

[1] We will assume that a verification involving a combination of the *Axiom of union* and the *Axiom of power set* guarantees that $\prod_\alpha S_\alpha$ is indeed a "set".

$y_\alpha = f(\alpha)$, then we can express it in one of the forms

$$u = \{f(\alpha) : \alpha \in I\} = \{y_\alpha : \alpha \in I\} = <y_\alpha>_{\alpha \in I}$$

or we can write it more simply as $\{y_\alpha\} = < y_\alpha >$. If $\beta \in I$, the set, S_β, is called the

β^{th} *factor*

of the Cartesian product, $\prod_\alpha S_\alpha$, while y_β is called the

β^{th} *coordinate*

of the element, $< y_\alpha >_{\alpha \in I}$

For example, if $I = \{1, 2, 3\}$

$$\textstyle\prod_{\alpha \in I} S_\alpha = \{(a, b, c) : a \in S_1, b \in S_2, c \in S_3\} = S_1 \times S_2 \times S_3$$

The size of $\prod_{\alpha \in I} S_\alpha$ depends on the size of the respective sets, S_α, and the size of the index set.

If we are given an indexed family of sets, say $\{S_\alpha : \alpha \in J\}$, where all factors, S_α, represent the same set, S, the reader should be familiar with the following notation:

$$S^J = \textstyle\prod_{\alpha \in J} S$$

The expression, S^J, is normally interpreted as meaning

"*the set of all functions mapping J into S*"

For example, if $J = \mathbb{N}$ and $S_\alpha = \mathbb{R}$ for all $\alpha \in J$, then

$$\mathbb{R}^{\mathbb{N}} = \textstyle\prod_{\alpha \in \mathbb{N}} \mathbb{R} = \{< x_\alpha >_{\alpha \in \mathbb{N}} : x_\alpha \in \mathbb{R}\}$$

This product can also be expressed as $\mathbb{R}^{\mathbb{N}} = \prod_{\alpha \in \mathbb{N}} \mathbb{R}_\alpha$. It thus represents all countably infinite sequences of real numbers, $\{a_0, a_1, a_2, a_3, \ldots\}$, or equivalently, the set of all functions mapping $\mathbb{N}$ into $\mathbb{R}$. Another example is, $\mathbb{R}^{\mathbb{R}}$, which represents the set of all functions mapping $\mathbb{R}$ into $\mathbb{R}$. [2]

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[2] Since $\mathbb{R}$ is not normally viewed as an index set, the expression, $\mathbb{R}^{\mathbb{R}}$, is usually the preferred form to represent the family of all functions mapping $\mathbb{R}$ into $\mathbb{R}$ although, it may sometimes be more convenient to use the form $\prod_{i \in \mathbb{R}} \mathbb{R}_i$.

A few words on the definition of Cartesian product. In our definition of Cartesian product we refer to an indexed family, $\{S_\alpha : \alpha \in I\}$, of sets. We didn't state explicitly that each one of these is non-empty. Should we include this requirement in the definition? What happens if, say $S_\beta = \varnothing$, for some $\beta \in I$? Since there is nothing in S_β, then there cannot exist a function which will map β to some element in S_β and so the cautious reader will conclude that $\prod_\alpha S_\alpha$ must be empty. This is not catastrophic, since to say the product is non-empty is to assume that each factor is non-empty. So we can leave it as is.

On the other hand, if no S_α is empty, are we guaranteed that there exists at least one function, $f : I \to \cup_{\alpha \in I} S_\alpha$ (in our mathematical universe of sets), such that f assigns to each β in I a particular element in its corresponding set S_β? If so, is there a way to decide which element should be selected by f? The assumption that at least one such function exists invokes the statement in the Axiom of choice:

> "Given any set $\mathscr{A}$ of non-empty sets, there is a rule f which associates to each set A in $\mathscr{A}$ some element $a \in A$".

So the Axiom of choice grants us permission to assume that at least one function f will select a point $f(\beta) = m_\beta$ in S_β for us. However, other than this guarantee that at least one f exists, we have no way of ever determining what that function f is. We are assuming the existence of a mathematical entity we will never ever see. Invoking the Axiom of choice is not ideal, but it is the best we can do. For most people, the fact that the Cartesian product of non-empty sets is non-empty is obvious and is simply not worth losing any sleep over it. Throughout this section, we will, as a rule, assume the Axiom of choice holds true, and not point out it's application at each place it is involved, unless it is of particular interest to do so (as it, sometimes, is).

In this book, we declare that the "Cartesian product of topological spaces, $S = \prod_{i \in I} S_i$", we will always mean "the *non-empty* Cartesian product, S, of spaces."

A few set-theoretic properties of Cartesian products.

We have yet to topologize Cartesian products of spaces. But before we do so, we present a few of the Cartesian product's most fundamental properties.

Definition 7.2 Let $S = \prod_{i\in I} S_i$ be a Cartesian product of sets. The family of functions $\{\pi_i : i \in I\}$ where $\pi_j : \prod_{i\in I} S_i \to S_j$, is defined as

$$\pi_j(< x_i >_{i\in I}) = x_j$$

We will refer to π_j as being the j^{th} *projection* which maps $\prod_{i\in I} S_i$ *onto* S_j.

For example, consider the product $A\times B = \{(a, b) : a \in A, b \in B\}$ (with only two factors A and B) where $\pi_1[A \times B] = A$ and $\pi_2[A \times B] = B$. Then $\pi_1[\{a\} \times B] = \{a\}$ and $\pi_2[A \times \{b\}] = \{b\}$. Also note that

$$\pi_1^{\leftarrow}(c) = \{(a, b) \in A \times B : a = c\} = \{c\} \times B$$

Theorem 7.3 Let $\{S_i : i \in I\}$ and $\{T_i : i \in I\}$ be two families of sets and $S = \prod_{i\in I} S_i$ and $T = \prod_{i\in I} T_i$ be their corresponding Cartesian products.

(a) If $\prod_{i\in I} S_i \subseteq \prod_{i\in I} T_i$, then $S_i \subseteq T_i$ for each $i \in I$.

(b) For U_i, V_i, both subsets of S_i,

$$\prod_{i\in I} U_i \;\cap\; \prod_{i\in I} V_i = \prod_{i\in I}(U_i \cap V_i)$$

Proof :(a) Let $j \in I$. Since the j^{th} projection map, π_j, is onto the $(S_j)^{\text{th}}$ factor,

$$S_j = \pi_j\left(\prod_{i\in I} S_i\right) \subseteq \pi_j\left(\prod_{i\in I} T_i\right) = T_j$$

The proof of part (b) are left for the reader.

In particular, it is always true that

$$[A \times B] \;\cap\; [C \times D] \;=\; [A \cap C] \times [B \cap D]$$

7.2 Topologizing the Cartesian product of topological spaces.

Taking Cartesian products of large numbers of topological spaces is a powerful way to construct new topological spaces from "old" ones. There is more than one topology for us to choose from, some of which will eventually be more useful than others, depending on the context. For finite products there is one which most users would consider as being the most intuitive and as being the "standard" one. For infinite products the choice of topology may depend on which one appears to be more useful in the situation at hand.

The third example on page 95, illustrates how one might proceed to topologize the Cartesian product of two spaces (S, τ_S) and (T, τ_T). In that example, the topology, τ, on $S \times T$ was generated by choosing, as subbase for τ, the family of sets

$$\mathscr{S} = \{\pi_S^{\leftarrow}[U] : U \in \tau_S\} \cup \{\pi_T^{\leftarrow}[V] : V \in \tau_T\}$$

where $\pi_S : S \times T \to S$ and $\pi_T : S \times T \to T$ are projection maps. Recall that $\pi_S(a, b) = a$ for all $b \in T$ and $\pi_T(a, b) = b$ for all $a \in S$ and so $\pi_S^{\leftarrow}(a) = \{a\} \times T$ and $\pi_T^{\leftarrow}(b) = S \times \{b\}$. So the set of all finite intersections of the elements in $\mathscr{S}$ forms a base, $\mathscr{B}$, for this topology. Thus, the base elements of τ are of the form

$$\begin{aligned} \pi_S^{\leftarrow}[U] \cap \pi_T^{\leftarrow}[V] &= (U \times T) \cap (S \times V) \\ &= (U \cap S) \times (V \cap T) \\ &= U \times V \end{aligned}$$

where $U \in \tau_S$ and $V \in \tau_T$. That is, $\mathscr{B} = \{U \times V : U \in \tau_S, V \in \tau_T\}$.

Note that, in this particular example, the topology constructed for $S \times T$ is the

"*weak topology induced by the projection functions* $\{\pi_S, \pi_T\}$"

This "weak topology" on $S \times T$ comes with the guarantee of continuity for each projection map, π_α, on $S \times T$. This motivates the choice of the "standard topology" for arbitrary products described in the following definition.

Definition 7.4 Let $\{(S_\alpha, \tau_\alpha) : \alpha \in I\}$ be an indexed family of non-empty topological spaces and

$$S = \prod_{\alpha \in I} S_\alpha$$

be the Cartesian product of these spaces. Let $\{\pi_\alpha : \alpha \in I\}$ be the family of the associated projection maps

$$\pi_\beta : \prod_{\alpha\in I} S_\alpha \to S_\beta$$

We define the *product topology* or the *standard topology* as being the weak topology on $\prod_{\alpha\in I} S_\alpha$ generated by the family of functions $\{\pi_\alpha : \alpha \in I\}$. The Cartesian product of topological spaces, when equipped with this weak topology, is referred to as a *product space.* The family of all sets of the form,

$$\{\pi_\alpha^{\leftarrow}[U] : U \in \tau_\alpha\}$$

is the subbase, $\mathscr{S}$, for the product space, while each element of the base of open sets, $\mathscr{B}$, is of the form

$$\cap_{\alpha\in F}\{\pi_\alpha^{\leftarrow}[U_\alpha] : U_\alpha \in \tau_\alpha\}$$

where F is a finite subset of I.

Since $\pi_\alpha^{\leftarrow}[U_\alpha] \cap \pi_\alpha^{\leftarrow}[V_\alpha] = \pi_\alpha^{\leftarrow}[U_\alpha \cap V_\alpha]$, we can assume that all the α's in the finite set, F, are distinct. This assumption does not alter the definition of *product topology.* Also, $\pi_\beta^{\leftarrow}[U_\beta]$ is a subset of $\prod_{\alpha\in I} S_\alpha$ where, if $\alpha \neq \beta$, the α^{th} factor is, S_α, itself, and only the β^{th} factor is U_β. Hence, for any open base element, every factor is S_α itself except for finitely many factors, U_α, as proper subsets of S_α.

It is also worth noting that the product topology is the absolute smallest topology on $S = \prod_{\alpha\in I} S_\alpha$ which guarantees that each and every projection map in

$$\{\pi_\alpha : \prod_{\alpha\in I} S_\alpha \to S_\alpha\}$$

is continuous on its domain, S.

Finally, note that. . . ,

> "*the product topology on the Cartesian product depends on the topology which has been defined on each of its factors*".

It does not depend on some topology defined on the index set, I. There is no topology defined on the index set.

Projection maps are open.

In the following theorem we will show that, for a product space $\prod_{\alpha\in I} S_\alpha$, each map, π_α, is an open map. This is important to keep this in mind.

Theorem 7.5 Given a product space, $S = \prod_{\alpha \in I} S_\alpha$, the projection map,

$$\pi_\beta : S \to S_\beta$$

is an open map.

Proof: Given: A product space, $S = \prod_{\alpha \in I} S_\alpha$, and a projection map, $\pi_\beta : S \to S_\beta$.

Since functions respect arbitrary unions, it suffices to show that the projection map sends basic open sets to open sets.

Let

$$V = \cap\{\pi^{\leftarrow}_{\alpha_i}[U_{\alpha_i}] : i = 1, 2, \ldots, k\}$$

be a basic open set in S. It suffices to show that $\pi_\beta[V]$is open in S_β.

If $\beta = \alpha_j$ then $\pi_\beta[V] = \pi_\beta[\cap\{\pi^{\leftarrow}_{\alpha_i}[U_{\alpha_i}] : i = 1, \ldots, k\}] = U_{\alpha_j}$.

If, for $i = 1, \ldots, k$, $\beta \neq \alpha_i$ then $\pi_\beta[V] = \pi_\beta[\cap\{\pi^{\leftarrow}_{\alpha_i}[U_{\alpha_i}] : i = 1, \ldots, k\}] = S_\beta$.

In both cases, $\pi_\beta[V]$ is open in S_β, so π_β is an open map, as required.

Having now convinced ourselves that whenever a Cartesian product is equipped with the product topology the projections maps are open maps, this does allow us to conclude that that they are “closed maps”. In general, they are not!

Theorem 7.6 Let $\{(S_\alpha, \tau_\alpha) : \alpha \in I\}$ be an indexed family of non-empty topological spaces and $\prod_{\alpha \in I} S_\alpha$ be the Cartesian product space equipped with the product topology, τ. For each $\alpha \in I$, S_α has an open base represented by, $\mathscr{B}_\alpha$. Then the set

$$\mathscr{B}^* = \{\ \cap_{\alpha \in F}\{\pi^{\leftarrow}_\alpha[B_\alpha] : B_\alpha \in \mathscr{B}_\alpha\}\ \}_{\alpha \in I}$$

where F is finite, forms a base for τ.

Proof: Let $V = \cap_{\alpha\in F}\{\pi_\alpha^{\leftarrow}[U_\alpha] : U_\alpha \in \tau_\alpha\}$ (with F finite) be an open base element of the product topology, τ. Suppose $<x_\alpha>_{\alpha\in I} \in V$. Then, for each $\alpha \in F$, there exists an open base element, $B_\alpha \in \mathscr{B}_\alpha$, such that $x_\alpha \in B_\alpha \subseteq U_\alpha$. Then

$$<x_\alpha>_{\alpha\in I} \in \cap_{\alpha\in F}\{\pi_\alpha^{\leftarrow}[B_\alpha] : B_\alpha \in \mathscr{B}_\alpha\} \subseteq V$$

Then every open base element of τ is the union of elements from $\mathscr{B}^*$. So $\mathscr{B}^*$ forms a base for the product topology.

The "box topology" on a Cartesian product.

The product topology is not the only topology we can define on (infinite) products of topological spaces. Some readers may have noticed that, for an infinite product, $S = \prod_{\alpha\in I} S_\alpha$, the collection

$$\mathscr{B}^* = \cap_{\alpha\in I}\{\pi_\alpha^{\leftarrow}[U_\alpha] : U_\alpha \in \tau_\alpha\}$$

(observing that the intersection is of *infinitely* many sets) satisfies the "base property". Hence, this set will be a base for a topology τ^* which is different from the product topology, τ, on S. In the literature, it is referred to as the

"*box topology* on the product $\prod_{\alpha\in I} S_\alpha$"

where S_α is a topological space, for each $\alpha \in I$.

Suppose

$$<x_\alpha>_{\alpha\in I} \in \ B = \cap_{\alpha\in F}\{\pi_\alpha^{\leftarrow}[U_\alpha] : U_\alpha \in \tau_\alpha\}$$

where F is a finite subset of I. Then

$$<x_\alpha>_{\alpha\in I} \in \ \cap_{\alpha\in I}\{\pi_\alpha^{\leftarrow}[U_\alpha] : U_\alpha \in \tau_\alpha\} \ \subseteq \ B$$

So every open base element for the product topology is open in the box topology. That is, $\tau_{\text{prod}} \subseteq \tau_{\text{box}}$.

The "box topology" is strictly finer than the "product topology". To see this, consider, for example, a basic open neighborhood, $\prod_{\alpha\in I} U_\alpha$ of the point $<x_\alpha>_{\alpha\in I}$ with respect to the box topology on $\prod_{\alpha\in I} S_\alpha$, where $U_\alpha \neq S_\alpha$ for all $\alpha \in I$. If B_1 is a basic open neighborhood of $< x_\alpha >_{\alpha\in I}$ in the product space S, all but finitely many factors of B_1 are equal to $[0, 1]$. So B_1 cannot be contained in B. That is, the box topology has sets which do not belong to the product topology.

Note, however, that, for the Cartesian product of *finitely* many spaces, the product topology and the box topology are equivalent topologies.

The "uniform topology" on a Cartesian product $\prod_{j\in J}[a,b]$.

We take the opportunity to introduce another topology on a product $\prod_{j\in J}[a,b]$.

Let $S = \prod_{j\in J}[a,b]$.

For $< x(j) >_{j\in J}$ and $< y(j) >_{j\in J}$ in $\prod_{j\in J}[a,b]$ we define,

$$\rho(< x(j) >_{j\in J},\ < y(j) >_{j\in J}) = \sup\{|x(j) - y(j)| : j \in J\}$$

Verification that this is a valid metric on S is left to the reader.

The metric, ρ, is referred to as the

uniform metric on $\prod_{j\in J}[a,b]$

The topology, τ_ρ, corresponding to the uniform metric is called the

uniform topology induced by ρ

The metric topology, τ_ρ, induced by ρ on S, is not equivalent to the product topology, τ_{prod}, on S. To see this, let $< 0(j) >_{j\in J} \in S$ and $\varepsilon = 1/10$. Suppose $B_{1/10}(< 0(j) >)$ is a basic open neighborhood of $< 0(j) >$ in τ_ρ. Then

$$\begin{aligned} B_{1/10}(< 0(j) >) &= \{< y(j) > \in S : \rho(< 0(j) > - < y(j) >) < 1/10\} \\ &= \{< y(j) > \in S : \rho(< 0 - y(j) >) < 1/10\} \\ &= \{< y(j) > \in S : \sup_{j\in J} |y(j)| < 1/10\} \end{aligned}$$

No basic open neighborhood of $< 0(j) >_{j\in J}$ with respect to the product topology, can "fit" inside $B_{1/10}(< 0(j) >_{j\in J})$, since all but finitely many of its factors are equal to $[0,1]$. So $B_{1/10}(< 0(j) >)$ is not open with respect to the product topology on S. So $\tau_\rho \not\subseteq \tau_{\text{prod}}$.

Example 1. Suppose $a = (a_1, a_2, a_3) = (1, 4, 3)$ and $b = (b_1, b_3, b_3) = (-1, 0, -3)$.

Then

$$\begin{aligned}\rho(a,b) &= \sup\{|a_i - b_i| : i = 1,2,3\}\\ &= \sup\{|a_1 - b_1|, |a_2 - b_2|, |a_3 - b_3|\}\\ &= \sup\{2,4,6\}\\ &= 6\end{aligned}$$

Example 2. Suppose the two functions, $f(j) = 1/(j^2+1)$ and $g(j) = \arctan(j)$, are viewed as elements of $\prod_{j\in\mathbb{Z}}[-3,3]$. Then

$$\begin{aligned}\rho(< f(j) >_{j\in\mathbb{Z}}, < g(j) >_{j\in\mathbb{R}}) &= \sup\{|f(j) - g(j)| : j \in \mathbb{Z}\}\\ &= \sup\{|1/(j^2+1) - \arctan(j)| : j \in \mathbb{Z}\}\\ &= \pi/2\end{aligned}$$

A word of caution: Theorems in which the Cartesian product equipped with the product topology is involved may not hold true if that product is equipped with the uniform topology.

Example 3. For each $j \in \mathbb{N}$, let

$$f_j(x) = \sin(x)/(j^2+1)$$

a function mapping $\mathbb{R}$ into $[-1,1]$. Let $\mathscr{F} = \{f_j : j \in \mathbb{N}\}$. Define the function, $e_{\mathscr{F}} : \mathbb{R} \to \prod_{j\in\mathbb{N}}[-1,1]$, as

$$e_{\mathscr{F}}(x) = < f_j(x) >_{j\in\mathbb{N}} \in \prod_{j\in\mathbb{N}}[-1,1]$$

Then $e_{\mathscr{F}}[\mathbb{R}] \subseteq \prod_{j\in\mathbb{N}}[-1,1]$. We can equip $\prod_{j\in\mathbb{N}}[-1,1]$ with the uniform metric topology τ_ρ. Consider $< 0_j >_{j\in\mathbb{N}} \in \prod_{j\in\mathbb{N}}[-1,1]$.

Does $< 0_j >_{j\in\mathbb{N}}$ belong to $\mathrm{cl}[e_{\mathscr{F}}[\mathbb{R}]\,]$ with respect to τ_ρ?

For $\varepsilon_j = \varepsilon > 0$,

$$\begin{aligned}\rho(< \varepsilon_j >_{j\in\mathbb{N}}, < 0_j >_{j\in\mathbb{N}}) &= \rho(< \varepsilon_j - 0_j >_{j\in\mathbb{N}})\\ &= \sup\{|\varepsilon_j| : j \in \mathbb{N}\}\\ &= \varepsilon\end{aligned}$$

Are there values of x such that $e_{\mathscr{F}}(x) \in B_\varepsilon(< 0_j >_{j\in\mathbb{N}})$?

Then, for $x \in \mathbb{R}$,

$$\begin{aligned}\rho(e_{\mathscr{F}}(x), < 0_j >_{j\in\mathbb{N}}) &= \rho(< f_j(x) >_{j\in\mathbb{N}}, < 0_j >_{j\in\mathbb{N}})\\ &= \rho(< f_j(x) - 0_j >_{j\in\mathbb{N}})\\ &= \sup\{|f_j(x) - 0_j| : j \in \mathbb{N}\}\\ &= \sup\{|\tfrac{\sin(x)}{j^2+1}| : j \in \mathbb{N}\}\\ &= |\sin(x)| \quad \text{\scriptsize (}|f_j(x)|\text{\scriptsize is maximal when } j = 0.\text{\scriptsize)}\end{aligned}$$

For any $\varepsilon > 0$, for x such that $|\sin(x)| < \varepsilon$, $e_{\mathscr{F}}(x) \in B_\varepsilon(< 0_j >_{j\in\mathbb{N}})$.

So, for any $\varepsilon > 0$, $e_{\mathscr{F}}[\mathbb{R}] \cap B_\varepsilon(< 0_j >_{j\in\mathbb{N}}) \neq \varnothing$.

So $< 0_j >_{j\in\mathbb{N}}$ belongs to $\text{cl}[e_{\mathscr{F}}[\mathbb{R}]\,]$ with respect to τ_ρ.

Example 4. Consider the set, $[-1,1]^{\mathbb{R}}$, of all functions mapping $\mathbb{R}$ into $[-1,1]$ equipped with the usual topology. Express this set as the product $\prod_{x\in\mathbb{R}}[-1,1]_x$ (where $[-1,1]_x$ is the x^{th} factor of the product.)

Consider the subset,

$$A = \left\{ f \in [-1,1]^{\mathbb{R}} : \frac{-1}{x^2+1} < f(x) < \frac{1}{x^2+1}, x \in \mathbb{R} \right\}$$

of $\prod_{x\in\mathbb{R}}[-1,1]_x$.

a) If we equip $\prod_{x\in\mathbb{R}}[-1,1]_x$ with the box topology then the subset A is open, since

$$A = \cap_{x\in\mathbb{R}} \left\{ \pi_x^{\leftarrow}\left[\left(\frac{-1}{x^2+1}, \frac{1}{x^2+1}\right)\right] : \left(\frac{-1}{x^2+1}, \frac{1}{x^2+1}\right) \subseteq [-1,1]_x \right\}$$

b) On the other hand, suppose $\prod_{x\in\mathbb{R}}[-1,1]_x$ is equipped with the uniform norm topology, τ_ρ, where

$$\rho(f(x), g(x)) = \sup\{|f(x) - g(x)| : x \in [-1,1]\}$$

Consider the element $0(x)$ in A.

Suppose A is open in $\prod_{x\in\mathbb{R}}[-1,1]_x$ with respect to τ_ρ. There should be, for some ε_1, a basic open neighborhood $B_{\varepsilon_1}(0(x))$ of $0(x)$ (with respect to τ_ρ) so that $B_{\varepsilon_1}(0(x)) \subseteq A$.

$$\begin{aligned} B_{\varepsilon_1}(0(x)) &= \{f : \rho(f(x), 0(x)) < \varepsilon_1\} \\ &= \{f : \sup\{|f(x)| : x \in \mathbb{R}\} < \varepsilon_1\,\} \end{aligned}$$

Suppose $h(x) = \frac{\varepsilon_1}{2}$, on $\mathbb{R}$. Then, since $\sup\{|h(x) - 0(x)| : x \in \mathbb{R}\} = \frac{\varepsilon_1}{2} < \varepsilon_1$, then $h \in B_{\varepsilon_1}(0(x))$.

Then $\frac{\varepsilon_1}{2} < \frac{1}{x^2+1}$ for all x. But $\lim_{x\to\infty} \frac{1}{x^2+1} = 0$. Hence $B_{\varepsilon_1}(0(x))$ cannot be contained in A.

More topics involving the product topology.

The content of the remaining part of this chapter need not be covered immediately. The results may, however, be referred to quite a bit later on in the text. These can thus be studied at that point when there is more motivation to do so.

7.3 Topic: On interiors, closures and boundaries of product spaces.

It is sometimes useful to know how to handle the interior and the closure of a product. The interior of a product is well-behaved only in the case of a finite product space, while the closure of a product distributes nicely over its factors even for infinite products.

Theorem 7.7 Let $\{(S_\alpha, \tau_\alpha) : \alpha \in I\}$ be an indexed family of non-empty topological spaces and $S = \prod_{\alpha \in I} S_\alpha$ be a Cartesian product equipped with the product topology. Let $T = \prod_{\alpha \in I} A_\alpha$ where each A_α is a subspace of S_α. Then,

$$\text{cl}_S T = \prod_{\alpha \in I} \text{cl}_{S_\alpha} A_\alpha$$

Proof: The Cartesian product space $S = \prod_{\alpha \in I} S_\alpha$ and $A_\alpha \subseteq S_\alpha$ are given.

Let $T = \prod_{\alpha \in I} A_\alpha$.

We claim that $\prod_{\alpha \in I} \text{cl}_{S_\alpha} A_\alpha \subseteq \text{cl}_S T$.

Let $<x_\alpha> \in \prod_{\alpha \in I} \text{cl}_{S_\alpha} A_\alpha$. To show that $<x_\alpha> \in \text{cl}_S T$ we must show that any open neighborhood of $<x_\alpha>$ in S meets $\prod_{\alpha \in I} A_\alpha$.

Let $U = \prod_{\alpha \in I} U_\alpha$ be a basic open neighborhood of $<x_\alpha>$ in $\prod_{\alpha \in I} S_\alpha$.

Then, for each α, $x_\alpha \in \text{cl}_{S_\alpha} A_\alpha \cap U_\alpha$. Then $U_\alpha \cap A_\alpha \neq \varnothing$.

For each α, there exists $y_\alpha \in U_\alpha \cap A_\alpha$. Then

$$<y_\alpha> \in \prod_{\alpha \in I} A_\alpha \cap \prod_{\alpha \in I} U_\alpha = \prod_{\alpha \in I} A_\alpha \cap U$$

So

$$<x_\alpha> \in \text{cl}_S \prod_{\alpha \in I} A_\alpha$$

Then $\prod_{\alpha \in I} \text{cl}_{S_\alpha} A_\alpha \subseteq \text{cl}_S T$ as claimed.

We claim that $\text{cl}_S T \subseteq \prod_{\alpha \in I} \text{cl}_{S_\alpha} A_\alpha$.

Let $<x_\alpha> \in \text{cl}_S \prod_{\alpha \in I} A_\alpha$ and U_β be an open neighborhood of x_β in S_β. Then $\pi_\beta^{\leftarrow}[U_\beta]$ is open in $\prod_{\alpha \in I} S_\alpha$. The set $\pi_\beta^{\leftarrow}[U_\beta]$ contains some point $<y_\alpha> \in \prod_{\alpha \in I} A_\alpha$. Then $y_\beta \in U_\beta \cap A_\beta$. So $x_\beta \in \text{cl}_{S_\alpha} A_\beta$.

Therefore $<x_\alpha> \in \prod_{\alpha\in I}\mathrm{cl}_{S_\alpha}A_\alpha$.

We conclude that $\mathrm{cl}_S T \subseteq \prod_{\alpha\in I}\mathrm{cl}_{S_\alpha}A_\alpha$ as claimed.
We then have

$$\mathrm{cl}_S\prod_{\alpha\in I}A_\alpha = \prod_{\alpha\in I}\mathrm{cl}_{S_\alpha}A_\alpha$$

as required.

Note that the above statement holds true also if the product is equipped with the box topology.

Theorem 7.8 Let S and T be topological spaces which contain subsets A and B, respectively. Then,

$$\mathrm{int}_{S\times T}(A\times B) = \mathrm{int}_S A \times \mathrm{int}_T B$$

Proof : We are given the Cartesian product $M = S\times T$ and $A \subseteq S$, $B\subseteq T$.

Since $\mathrm{int}_S A$ and $\mathrm{int}_T B$ are both open in S and T, respectively, then $\mathrm{int}_S A \times \mathrm{int}_T B$ is open in M. Since $\mathrm{int}_S A \times \mathrm{int}_T B \subseteq A\times B$, then

$$\mathrm{int}_S A \times \mathrm{int}_T B \subseteq \mathrm{int}_M(A\times B)$$

Conversely, let $(a,b)\in \mathrm{int}_M(A\times B)$. We claim $(a,b)\in \mathrm{int}_S A\times \mathrm{int}_T B$. There is an M-open neighborhood, U, of (a,b) such that

$$(a,b)\in U \subseteq \mathrm{int}_M(A\times B) \subseteq A\times B$$

Then $U = \cup_{i\in I}(V_i\times W_i)$, where V_i is open in S and W_i is open in T. Let $j\in I$ such that $(a,b)\in V_j\times W_j\subseteq U\subseteq A\times B$. Then $a\in \mathrm{int}_S A$ and $b\in \mathrm{int}_T B$. Then $(a,b)\in \mathrm{int}_S A\times \mathrm{int}_T B$.

We conclude that $\mathrm{int}_M(A\times B)\subseteq \mathrm{int}_S A\times \mathrm{int}_T B$.

So $\mathrm{int}_M(A\times B) = \mathrm{int}_S A\times \mathrm{int}_T B$.

We should be careful not to generalize the statement in the theorem immediately (involving interiors) above to arbitrary products.

Consider, for example, $M = \prod_{n\in\mathbb{N}}\mathbb{R}$ equipped with the product topology, and the subset, $\prod_{n\in\mathbb{N}}(0,1)$, of M. Then $\prod_{n\in\mathbb{N}}\mathrm{int}_{\mathbb{R}}(0,1) = \prod_{n\in\mathbb{N}}(0,1)$.

On the other hand consider,

$$W = \text{int}_M \left[\prod_{n\in\mathbb{N}}(0,1)\right]$$

Let $U = \prod_{n\in\mathbb{N}} Y_n$ be a basic open subset (with respect to the product topology) of M, where $Y_n = \mathbb{R}$ for all but finitely many n's. Since $\prod_{n\in\mathbb{N}}(0,1)$ cannot contain even a single basic M-open set, $\prod_{n\in\mathbb{N}}(0,1)$ can have no interior.

We conclude that $\text{int}_M \left[\prod_{n\in\mathbb{N}}(0,1)\right] = \varnothing \neq \prod_{n\in\mathbb{N}}\text{int}_{\mathbb{R}}(0,1)$.

Example 5. Let S and T be topological spaces which contain subsets A and B, respectively. Verify that

$$\text{bd}_{S\times T}(A\times B) \;=\; [\text{cl}_S A \times\ \text{bd}_T B] \cup [\ \text{bd}_S A \times \text{cl}_T B]$$

Solution. See that

$$\begin{aligned}
\text{bd}_{S\times T}(A\times B) &= \text{cl}_{S\times T}(A\times B)\backslash\text{int}_{S\times T}(A\times B)\\
&= (\text{cl}_S A \times \text{cl}_T B)\backslash(\text{int}_S A \times \text{int}_T B)\\
&= [\text{cl}_S A\backslash\text{int}_S A \times \text{cl}_T B)] \cup [\text{cl}_S A \times \text{cl}_T B\backslash\text{int}_T B]\\
&= [\ \text{bd}_S A \times \text{cl}_T B] \cup [\text{cl}_S A \times\ \text{bd}_T B]
\end{aligned}$$

We are done with Example 5.

7.4 Topic: Countability properties involving product spaces.

In this text, when we say "product space", we will always mean the Cartesian product equipped with the product topology. The product topology is to be considered as the default topology unless stated otherwise.

In the following two theorems we investigate whether the countably properties carry over from factors to product and vice versa. Recall that a topological space, (S, τ_S), is *second countable* if it has a countable base for open sets. It is *first countable* if each point, $x \in S$, has a countable open neighborhood base at x.

Theorem 7.9 Let $\{(S_\alpha, \tau_\alpha) : \alpha \in \mathbb{N}\}$ be an indexed *countable* family of nonempty topological spaces and $S = \prod_{\alpha\in\mathbb{N}} S_\alpha$ be the corresponding product space.

(a) Then the product space, S, is second countable if and only if each S_α is second countable.

(b) Then the product space, S, is first countable if and only if each S_α is first countable.

Proof: We are given that $S = \prod_{\alpha \in \mathbb{N}} S_\alpha$ is a product space of countably many factors.

We will prove part (a).

($\Rightarrow$) Suppose S is second countable. We claim that each S_α is second countable.

Recall (from proposition 7.5) that the projection map π_α is continuous and open for each $\alpha \in \mathbb{N}$. Hence, by Theorem 6.12, given that π_α is continuous each $S_\alpha = \pi_\alpha[S]$ is second countable. We are done with this direction. (Note that the countability of the index set plays no role in this direction.)

($\Leftarrow$) We are given that, for each $\alpha \in \mathbb{N}$, (S_α, τ_α) is a second countable topological space, and $\prod_{\alpha \in \mathbb{N}} S_\alpha$ is a countable product equipped with the product topology, τ.

By hypothesis, for each $\alpha \in \mathbb{N}$, S_α has a countable base, $\mathscr{B}_\alpha$, of open sets. Then, by Theorem 7.6, the elements of τ of the form

$$\cap_{\alpha \in F} \{\pi_\alpha^{\leftarrow}[B_\alpha] : B_\alpha \in \mathscr{B}_\alpha\}$$

when gathered together form a base, $\mathscr{B}$, for open sets in τ. Since each $\mathscr{B}_\alpha$ is countable, then so is the subfamily

$$\{\pi_\alpha^{\leftarrow}[B_\alpha] : B_\alpha \in \mathscr{B}_\alpha, \alpha \in \mathbb{N}\}$$

of the base, $\mathscr{B}$, of S. Let $\mathscr{F} = \{F : F \subset \mathbb{N},\ F \text{ is finite}\}$. Note that the set, $\mathscr{F}$, has countably many elements. (See footnote[3]) (This is where the countable the number of factors is countable is taken into consideration.)

Since the cardinality of $\mathscr{F} \times \mathscr{B}_\alpha \times \mathbb{N}$ is $\aleph_0$, then the base

$$\mathscr{B} = \{\ \cap_{\alpha \in F} \{\pi_\alpha^{\leftarrow}[B_\alpha] : B_\alpha \in \mathscr{B}_\alpha\}\ \}_{F \text{ finite},\ \alpha \in \mathbb{N}}$$

is countable. This means $\prod_{\alpha \in \mathbb{N}} S_\alpha$ is second countable.

The proof of part (b) is similar and so is left as an exercise.

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[3] The number of finite subsets of $\mathbb{N}$ is countable: For each $n \in \mathbb{N}$, let $\mathscr{U}_n = \{A \in \mathscr{P}(\mathbb{N}) : \max A \leq n\}$, the set of all subsets of $\{0, 1, 2, \ldots, n\}$. Then, for each n, $|\mathscr{U}_n| = 2^{n+1}$. The countable union of countable sets is countable (See R. André, *Set theory: An introduction to Axiomatic Reasoning*), so $\cup_{n \in \mathbb{N}} \{\mathscr{U}_n\}$ is countable. If F is a finite subset of $\mathbb{N}$, $F \in \mathscr{U}_m$ for some m. Then $\cup_{n \in \mathbb{N}} \{\mathscr{U}_n\}$ contains all finite subsets of $\mathbb{N}$. Then the number of finite subsets of $\mathbb{N}$ is countable.

In the following analogous theorem, the "separable property" will carry over from factors to the product and vice-versa, provided the number of factors in the product is restricted to no more than $c = |\mathbb{R}| = 2^{\aleph_0}$ (the cardinality of $\mathbb{R}$). The provided proof is notationally involved, but exhibits an interesting technique.

Theorem 7.10 Let $\{(S_\alpha, \tau_\alpha) : \alpha \in I\}$ be a family of sets indexed by the set I, where $|I| \leq c = 2^{\aleph_0}$. Also, suppose that S_α is a non-singleton set, for all $\alpha \in I$. Let $S = \prod_{\alpha \in I} S_\alpha$, be the corresponding product space. Then S is separable if and only if each S_α is separable.

Proof: We are given that $S = \prod_{\alpha \in I} S_\alpha$ is a product space of non-empty non-singleton sets where $|\, I \,| \leq c = 2^{\aleph_0}$.

($\Rightarrow$) Suppose S is separable. Recall from the proof that appears on page 150, the projection map π_α is continuous for each $\alpha \in \mathbb{N}$. Hence by Theorem 6.11, the continuous image of a separable space is separable, so each $S_\alpha = \pi_\alpha[S]$ is separable. We are done with this direction. The cardinality of I is irrelevant for this direction.

($\Leftarrow$) Suppose each S_α is separable and $|I| \leq c$. Then $|I| \leq |\, [0,1] \,|$. So there is a one-to-one function $f : I \to [0,1]$ which maps I *into* $[0,1]$. We can then index the product space S with $f[I]$. That is, $S = \prod_{\alpha \in I} S_{f(\alpha)}$. So, if $x \in S$, then x is of the form $< x_{f(\alpha)} >_{\alpha \in I}$.

Since each factor of S is separable, each $S_{f(\alpha)}$ contains a countable dense subset

$$D_{f(\alpha)} = \{d_{{f(\alpha)}_n} : n = 1, 2, 3, \ldots, \}$$

We will now construct a countable subset, D^*, of S which is dense in S.

For each $k \in \mathbb{N} \setminus \{0\}$ we can construct in $[0,1]$ a finite sequence of pairwise disjoint closed intervals

$$\mathscr{A}_k = \{[a_j, b_j]_k : j \in \{1, 2, \ldots, k\} \}$$

such that $a_j, b_j \in \mathbb{Q} \cap [0,1]$ and $a_{j+1} > b_j \leq 1$.

For each k, if $\alpha \in I$, $f(\alpha) \in [a_j, b_j]_k$ for, *at most*, one $j \in \{1, 2, \ldots, k\}$.

See that $\mathscr{W} = \cup\{\mathscr{A}_k : k = 1, 2, 3, \ldots\}$ is a countable set of intervals. We can then index the elements of $\mathscr{W}$ as

$$\mathscr{W} = \{[a_j, b_j]_{k,n} : j \in \{1, 2, \ldots, k\},\ k \in \{1, 2, 3, \ldots, \},\ n \in \{1, 2, 3, \ldots, \} \}$$

where, for each n, $[a_j, b_j]_{k,n} = [a_j, b_j]_k \in \mathscr{A}_k$.

We define a function $h : \mathscr{W} \to \prod_{\alpha\in I} D_{f(\alpha)} \subseteq \prod_{\alpha\in I} S_{f(\alpha)}$ as follows:

$$h(\ [a_j, b_j]_{k,n}\ ) = < y_{f(\alpha)} >_{\alpha\in I} \ \text{ where } \begin{cases} y_{f(\alpha)} = d_{f(\alpha)_n} & \text{if } f(\alpha) \in [a_j, b_j]_{k,n} \\ y_{f(\alpha)} = d_{f(\alpha)_1} & \text{otherwise} \end{cases}$$

Is $h : \mathscr{W} \to S$ a well-defined function? Yes it is, since (as mentioned earlier) for each k, if $\alpha \in I$, $f(\alpha) \in [a_j, b_j]_k$ for, *at most*, one $j \in \{1, 2, \ldots, k\}$.

Also, note that every coordinate of $h(\ [a_j, b_j]_{k,n}\ ) = < y_{f(\alpha)} >_{\alpha\in I}$ is an element of $D_{f(\alpha)}$, which is dense a dense subset of $S_{f(\alpha)}$.

Let $D^* = h[\mathscr{W}]$.

Claim 1: That D^* is countable in S.

Since the domain of h, $\mathscr{W}$, is countable then so is its range, $D^* = h[\mathscr{W}]$, in $\prod_{\alpha\in I} S_{f(\alpha)}$, as claimed.

Claim 2: That D^* (the countable image of $\mathscr{W}$) is a dense subset of S. (If so, then S is separable.)

Let $B = \pi^{\leftarrow}_{\alpha_1}[U_{f(\alpha_1)}] \cap \cdots \cap \pi^{\leftarrow}_{f(\alpha_k)}[U_{f(\alpha_k)}]$ be a basic open neighborhood of a point in $S = \prod_{\alpha\in I} S_{f(\alpha)}$.

We need to verify that $B \cap D^* \neq \varnothing$.

Since $D_{f(\alpha)} = \{d_{f(\alpha)_n} : n = 1, 2, 3, \ldots,\}$ is dense in $S_{f(\alpha)}$, then for each $f(\alpha_i)$ there exists $d_{f(\alpha_i)_{n_i}} \in U_{f(\alpha_i)}$, for $i = 1\ldots, k$.

If $d_{f(\alpha_i)_m} \in S_{f(\alpha_i)}$, then

$$\pi_{f(\alpha_i)}{}^{\leftarrow}(d_{f(\alpha_i)_m}) = \{< y_{f(\alpha)} >:\ f(\alpha_i) \in [a_j, b_j]_{k,m}\} = h([a_j, b_j]_{k,m})] \in h[\mathscr{W}]$$

$$\pi_{f(\alpha_i)}{}^{\leftarrow}(d_{f(\alpha_i)_1}) = \{< y_{f(\alpha)} >:\ f(\alpha_i) \notin [a_j, b_j]_{k,m}\} = h([a_j, b_j]_{k,m}) \in h[\mathscr{W}]$$

Then, $h[\mathscr{W}] \cap \pi^{\leftarrow}_{\alpha_1}[U_{f(\alpha_1)}] \cap \cdots \cap \pi^{\leftarrow}_{f(\alpha_k)}[U_{f(\alpha_k)}] \neq \varnothing$. Then $B \cap D^* \neq \varnothing$, so D^* is dense in S, as claimed.

So $h[\mathscr{W}]$ is a countable dense subset of $\prod_{\alpha\in I} S_{f(\alpha)}$. Then $\prod_{\alpha\in I} S_{f(\alpha)}$ is separable, as required.

One should exercise caution when transferring a topological property possessed by all factors of a product to the product itself. The statements of the above two theorems were proven with particular restrictions on the number of factors.

7.5 Topic: On continuous functions involving product spaces.

The choice of the "product topology" on a Cartesian product, $\prod_{\alpha\in I} S_\alpha$, is due mostly to the fact that it guarantees the continuity on a product space of all its projection maps, $\{\pi_\alpha : \alpha \in I\}$. It is the smallest topology that will guarantee this.

Lemma 7.11 Let $S = \prod_{\alpha\in I} S_\alpha$ be a product space and (T, τ) be a space. Suppose

$$g : T \to \prod_{\alpha\in I} S_\alpha$$

is a function mapping the space T to the product space, S. For each $\alpha \in I$, let the function $f_\alpha : T \to S_\alpha$ be defined as,

$$f_\alpha = \pi_\alpha \circ g$$

Then

$$[\, g \text{ is continuous on } T\,] \;\Leftrightarrow\; [\, f_\alpha : T \to S_\alpha \text{ is continuous for all } \alpha\,]$$

Proof: Recall that, by definition of "product space", each π_α is guaranteed to be continuous on S.

($\Rightarrow$) If $g : T \to S$ is continuous, then, for all $\alpha \in I$, so is $(\pi_\alpha \circ g) : T \to S_\alpha$ (since, by hypothesis, π_α is continuous, and the composition of continuous functions was shown to be continuous).

($\Leftarrow$) We fix $\beta \in I$. We are given that $(\pi_\beta \circ g) : T \to S_\beta$ is continuous. We are required to show that g is continuous. If V_β is open in S_β, since $(\pi_\beta \circ g)$ is continuous, then

$$g^{\leftarrow}\left[\pi_\beta^{\leftarrow}[V_\beta]\right] = (\pi_\beta \circ g)^{\leftarrow}[V_\beta]$$

is open in $S = \prod_{\alpha\in I} S_\alpha$. Since $\prod_{\alpha\in I} S_\alpha$ is equipped with the product topology, $\pi_\beta^{\leftarrow}[V_\beta]$ is a subbase element for this topology. So g pulls back an open subbase element, $\pi_\beta^{\leftarrow}[V_\beta]$, to an open subset of $S = \prod_{\alpha\in I} S_\alpha$. By Theorem 6.4, g is continuous on T, as required.

Remark. In the above theorem, the fact that the Cartesian product, $S = \prod_{\alpha\in I} S_\alpha$, is equipped with the product topology is what guarantees that each π_α is continuous. If S is equipped with the box topology the theorem may not hold true. The following example illustrates how the above theorem may break down if $\prod_{\alpha\in I} S_\alpha$ is equipped with the box topology.

Example 6. Consider the Cartesian product $\prod_{n\in\mathbb{N}} \mathbb{R}$ and suppose $\mathbb{R}$ is equipped with the usual topology. Let $i : \mathbb{R} \to \mathbb{R}$ denote the identity function $i(x) = x$ (a continuous function on $\mathbb{R}$). Let $g : \mathbb{R} \to \prod_{n\in\mathbb{N}} \mathbb{R}$ be a function where, for each $n \in \mathbb{N}$,

$$(\pi_n \circ g)(x) = i(x)$$

(where π_n is the n^{th} projection map). If $\prod_{n\in\mathbb{N}} \mathbb{R}$ is equipped with the product topology we know, by the theorem above, that g must be continuous on $\mathbb{R}$.

Suppose that $\prod_{n\in\mathbb{N}} \mathbb{R}$ is equipped with the box topology. Show that g need not be continuous.

Solution: If $\prod_{n\in\mathbb{N}} \mathbb{R}$ is equipped with the box topology, consider the open subset

$$B = (-1, 1) \times (-1/2, 1/2) \times (-1/3, 1/3) \times (-1/4, 1/4) \cdots$$

with respect to the box topology. Suppose $g^{\leftarrow}[B]$ was open in $\mathbb{R}$. Then there would exist ε, such that $g[(-\varepsilon, \varepsilon)] \subseteq B$. Then $(\pi_n \circ g)[(-\varepsilon, \varepsilon)] = i[(-\varepsilon, \varepsilon)] = (-\varepsilon, \varepsilon) \subseteq (-1/n, 1/n)$ for all n. This can't possibly be. So $g^{\leftarrow}[B]$ is not open. So g is not continuous on $\mathbb{R}$.

Theorem 7.12 Let $\{S_\alpha\}_{\alpha\in I}$ and $\{T_\alpha\}_{\alpha\in I}$ be two sets of topological spaces (with identical index set, I) and let $\{f_\alpha\}_{\alpha\in I}$ be a family of functions, $f_\alpha : S_\alpha \to T_\alpha$. Then the function, $g : \prod_{\alpha\in I} S_\alpha \to \prod_{\alpha\in I} T_\alpha$, defined as

$$g(<x_\alpha>_{\alpha\in I}) \; = <f_\alpha(x_\alpha)>_{\alpha\in I}$$

is continuous if and only if f_α is continuous on S_α for each $\alpha \in I$.

Proof: Suppose $S = \prod_{\alpha\in I} S_\alpha$, $T = \prod_{\alpha\in I} T_\alpha$ and $\{f_\alpha\}_{\alpha\in I}$ is a family of functions, $f_\alpha : S_\alpha \to T_\alpha$. Let $\pi_{\beta_S} : S \to S_\beta$ and $\pi_{\beta_T} : T \to T_\beta$ be β^{th} projection maps. Let $g : \prod_{\alpha\in I} S_\alpha \to \prod_{\alpha\in I} T_\alpha$ be defined as

$g(<x_\alpha>_{\alpha\in I}) \;= \;<f_\alpha(x_\alpha)>_{\alpha\in I}$.

($\Leftarrow$) For this direction we are given that each function f_α is continuous on S_α. We want to show that the continuity of g follows.

To show continuity of g it will suffice to show that g pulls back subbase elements of $\prod_{\alpha\in I} T_\alpha$ to open sets in $\prod_{\alpha\in I} S_\alpha$. Let U_β be an open subset of T_β. Then $\pi^{\leftarrow}_{\beta_T}[U_\beta]$ is a subbase element for open sets in $T = \prod_{\alpha\in I} T_\alpha$.

By Lemma 7.11, if $\pi_\alpha \circ g$ is continuous for all α, then g is continuous. It then suffices to show that $g^{\leftarrow}[\pi^{\leftarrow}_{\beta_T}[U_\beta]]$ is open in S.
See that,

$$\begin{aligned}
\pi^{\leftarrow}_{\beta_S}[f^{\leftarrow}_\beta[U_\beta]] &= \pi^{\leftarrow}_{\beta_S}[\{x\in S_\beta : f_\beta(x)\in U_\beta\}]\\
&= \{<x_\alpha>_{\alpha\in I}\in S : f_\beta(x_\beta)\in U_\beta\}\\
&= \{<x_\alpha>_{\alpha\in I}\in S : \{f_\alpha(x_\alpha)\}\in \pi^{\leftarrow}_{\beta_T}[U_\beta]\}\\
&= \{<x_\alpha>_{\alpha\in I}\in S : g(<x_\alpha>_{\alpha\in I})\in \pi^{\leftarrow}_{\beta_T}[U_\beta]\}\\
&= g^{\leftarrow}\left[<x_\alpha>_{\alpha\in I}\in S : \{x_\alpha\}\in \pi^{\leftarrow}_{\beta_T}[U_\beta]\}\right]\\
&= g^{\leftarrow}[\pi^{\leftarrow}_{\beta_T}[U_\beta]]
\end{aligned}$$

so,

$$g^{\leftarrow}[\pi^{\leftarrow}_{\beta_T}[U_\beta]] = \pi^{\leftarrow}_{\beta_S}[f^{\leftarrow}_\beta[U_\beta]]$$

Since both π_{β_S} and f_β are continuous, then the right-hand side is open; so $g^{\leftarrow}[\pi^{\leftarrow}_{\beta_T}[U_\beta]]$ is open. So g is continuous, as required for $\Leftarrow$.

($\Rightarrow$) Suppose $g : \prod_{\alpha\in I} S_\alpha \to \prod_{\alpha\in I} T_\alpha$ is continuous, where

$$g(<x_\alpha>_{\alpha\in I}) \;= \;<f_\alpha(x_\alpha)>_{\alpha\in I}$$

We are required to show that each $f_\alpha : S_\alpha \to T_\alpha$ is continuous.

Then for each β, $(\pi_\beta \circ g)(<x_\alpha>_{\alpha\in I}) = \pi_\beta(<f_\alpha(x_\alpha)>_{\alpha\in I}) = f_\beta(x_\beta) \in T_\beta$. Since both π_β and g are continuous on their domain, then so is f_β on S_β, as required. This proves $\Rightarrow$.

We expect that, *if two product spaces, S and T, have corresponding factors which are homeomorphic, then S and T are homeomorphic.* In the proof of the following theorem, we confirm this by explicitly exhibiting the required homeomorphism.

Theorem 7.13 Let $S = \prod_{\alpha\in I} S_\alpha$ and $T = \prod_{\alpha\in I} T_\alpha$ be two product spaces. Suppose that, for each $\alpha \in I$, S_α and T_α are homeomorphic. Then the spaces S and T are homeomorphic.

Proof: For each $\alpha \in I$, let $f_\alpha : S_\alpha \to T_\alpha$ be a homeomorphism mapping S_α onto T_α.

Let $g : \prod_{\alpha\in I} S_\alpha \to \prod_{\alpha\in I} T_\alpha$ be defined as

$$g(<x_\alpha>_{\alpha\in I}) = <f_\alpha(x_\alpha)>_{\alpha\in I} \in \prod_{\alpha\in I} T_\alpha$$

Since each f_α is continuous and one-to-one, then so is g (by Theorem 7.12 above).

We claim that g is open.
By hypothesis, each f_α is an open map. Let $\pi^{\leftarrow}_{\alpha_1}[U_{\alpha 1}] \cap \pi^{\leftarrow}_{\alpha_2}[U_{\alpha 2}] \cdots \cap \pi^{\leftarrow}_{\alpha_k}[U_{\alpha_k}]$ be an open base element in $\prod_{\alpha\in I} S_\alpha$. Since g is one-to-one, see that

$$\begin{aligned} g\left[\,\pi^{\leftarrow}_{\alpha_1}[U_{\alpha_1}] \cap \pi^{\leftarrow}_{\alpha_2}[U_{\alpha_2}] \cdots \cap \pi^{\leftarrow}_{\alpha_k}[U_{\alpha_k}]\,\right] & \\ &= g\left[\pi^{\leftarrow}_{\alpha_1}[U_{\alpha_1}]\right] \cap g\left[[\pi^{\leftarrow}_{\alpha_2}[U_{\alpha_2}]\right] \cdots \cap g\left[[\pi^{\leftarrow}_{\alpha_k}[U_{\alpha_k}]\right] \\ &= f_{\alpha_1}\left[\pi^{\leftarrow}_{\alpha_1}[U_{\alpha_1}]\right] \times \cdots \times f_{\alpha_k}\left[\pi^{\leftarrow}_{\alpha_k}[U_{\alpha_k}]\right] \end{aligned}$$

With all other factors equal to T_γ.

Since each f_{α_i} is open RHS is open in $\prod_{\alpha\in I} T_\alpha$.

Since the right-hand side is open, then g is open. So g is a homeomorphism.

We have shown that $g(<x_\alpha>_{\alpha\in I}) = <f_\alpha(x_\alpha)>_{\alpha\in I}$ is the required homeomorphism mapping S onto T.

We now show that every product space contains a homeomorphic copy of each of its factors.

Theorem 7.14 Let $S = \prod_{\alpha\in I} S_\alpha$ be a product space. Then, for each $\alpha \in I$, S contains a subspace which is a homeomorphic copy of S_α.

Proof: Let $\beta \in I$. For $\alpha \neq \beta$, we choose and fix $k_\alpha \in S_\alpha$. Define

$$T = \prod_{\alpha\in I} T_\alpha = \{<x_\alpha>_{i\in I} : T_\beta = S_\beta;\ \text{if } \alpha \neq \beta \text{ let } T_\alpha = \{k_\alpha\}\}$$

So every T is a subspace of S. We claim that T is homeomorphic to S_β.

We define $g : S_\beta \to T$ as $g(x) = <x_\alpha>_{\alpha\in I}$, where $x_\beta = x$, and if $\alpha \neq \beta$, $x_\alpha = k_\alpha$. Then g maps S_β one-to-one and onto $T \subseteq S$. Then the function, $(\pi_\beta \circ g) : S_\beta \to S_\beta$ is the continuous identity map onto

S_β. Invoking Lemma 7.11, we conclude that g is continuous on S_β. So g embeds S_β in the proper subset T of $\prod_{\alpha\in I} S_\alpha$ and so embeds S_β in $\prod_{\alpha\in I} S_\alpha$, as required.

We now study a metric, ρ, on a product space, $\prod_{n\in\mathbb{N}} S_n$, whose factors, (S_n, ρ_n), are metric spaces.

Example 7. Let $\{(S_n, \rho_n) : n \in \mathbb{N}\}$ be a countable family of metric spaces. Suppose that, for each $n \in \mathbb{N}$, $\sup\{\rho_n(x, y) : x, y \in S_n\} \leq 1$. Let

$$S = \prod_{n\in\mathbb{N}} S_n$$

the set of all sequences, $< x_i >_{i\in\mathbb{N}}$, with $x_n \in S_n$. If $f, g \in S$ then

$$f = < f(n) >_{n\in\mathbb{N}}, \qquad g = < g(n) >_{n\in\mathbb{N}}$$

Let $\rho : S \times S \to \mathbb{R}$ be defined as

$$\rho(f, g) = \sum_{n\in\mathbb{N}} \frac{\rho_n(\, f(n), \, g(n)\,)}{2^n}$$

Verify that $\rho : S \times S \to \mathbb{R}$ is a metric on the product space, S. Furthermore, verify that the topology on $S = \prod_{n\in\mathbb{N}} S_n$ derived by the metric, ρ, is, in fact, the product topology.

Solution: We are given that $S = \prod_{n\in\mathbb{N}} S_n$ where each S_n is a metric space and, for each n,

$$\sup\{\rho_n(x, y) : x, y \in S_n\} \leq 1$$

That is, the distance between any two elements of S_n does not exceed 1.

We claim that $\rho : S \times S \to \mathbb{R}$ thus defined, is a valid metric on S.

For $f, g \in S$ let $f = < f(n) >_{n\in\mathbb{N}}$ and $g = < g(n) >_{n\in\mathbb{N}}$.

- For M1, $\rho(f, g) = \sum_{n\in\mathbb{N}} \frac{\rho_n(\, f(n), g(n)\,)}{2^n} \leq \sum_{n\in\mathbb{N}} \frac{1}{2^n} = 2 \geq 0$.

  If $\rho(f, g) = \sum_{n\in\mathbb{N}} \frac{\rho_n(\, f(n), g(n)\,)}{2^n} = 0$, then $\rho_n(\, f(n), \, g(n)\,) = 0$ for all n.
  Since $f = < f(n) >_{n\in\mathbb{N}}$ and $g = < g(n) >_{n\in\mathbb{N}}$ then $f = g$.

- For M2, $\rho(f, g) = \sum_{n\in\mathbb{N}} \frac{\rho_n(\, f(n), g(n)\,)}{2^n} = \sum_{n\in\mathbb{N}} \frac{\rho_n(\, g(n), f(n)\,)}{2^n} = \rho(g, f)$.

- For M3, we compare $\rho(f,h)$ and $\rho(f,g)+\rho(g,h)$. Since, for each n, $\rho_n(\,f(n),\,h(n)\,) \leq \rho_n(\,f(n),\,g(n)\,) + \rho_n(\,g(n),\,h(n)\,)$, then

$$\begin{aligned}
\rho(f,h) &= \sum_{n\in\mathbb{N}} \frac{\rho_n(f(n),h(n))}{2^n} \\
&\leq \sum_{n\in\mathbb{N}} \frac{\rho_n(f(n),g(n)) + \rho_n(g(n),h(n))}{2^n} \\
&= \sum_{n\in\mathbb{N}} \left[\frac{\rho_n(f(n),g(n))}{2_n} + \frac{\rho_n(g(n),h(n))}{2^n}\right] \\
&= \sum_{n\in\mathbb{N}} \frac{\rho_n(f(n),g(n))}{2^n} + \sum_{n\in\mathbb{N}} \frac{\rho_n(g(n),h(n))}{2^n} \qquad \text{(All terms are positive)} \\
&= \rho(f,g)+\rho(g,h)
\end{aligned}$$

So ρ is a valid metric on $S = \prod_{n\in\mathbb{N}} S_n$, as claimed.

We will now compare the metric topology on (S,ρ) to the product topology on S.

Let

$$f = <f(n)>_{n\in\mathbb{N}} \in\ S$$

and

$$V = B_{1/2^p}(f)$$

be an open ball in S with respect to ρ.

Claim: We claim that V is open in S with respect to the product topology.

Proof of claim: Let

$$U = \{g \in S : g = <g(n)>,\ \ \rho(f(n),g(n)) < 2^{-p-n-2} \text{ for } n \geq p+2\}$$

Note that U is a union of open base elements and so is an open subset of S with respect to the product topology.

It suffices to show that $U \subset V$. If $y \in U$, then

$$\begin{aligned}
\rho(f,g) &= \sum_{n\in\mathbb{N}} \frac{\rho_n(f(n),g(n))}{2^n} \\
&= \left(\frac{\rho_0(f(0),g(0))}{2^0} + \frac{\rho_1(f(1),g(1))}{2^1} + \cdots + \frac{\rho_{p+2}(f(p+2),g(p+2))}{2^{p+2}}\right) \\
&\quad + \frac{\rho_{p+3}(\,f(p+3),\,g(p+3)\,)}{2^{p+3}} + \cdots \\
&< \left(\frac{2^{-p-0-2}}{2^0} + \frac{2^{-p-1-2}}{2^1} + \frac{2^{-p-2-2}}{2^2} + \cdots + \frac{2^{-p-(p+2)-2}}{2^{p+2}}\right) \\
&\quad + \frac{1}{2^{p+3}} + \frac{1}{2^{p+4}} + \cdots + \\
&= \left(\frac{1}{2^{p+2}} + \frac{1}{2^{p+4}} + \frac{1}{2^{p+6}} + \cdots + \frac{1}{2^{3p+6}}\right) + \left(\frac{1}{2^{p+3}} + \frac{1}{2^{p+4}} + \cdots +\right) \\
&< \frac{1}{2^{p+1}} + \frac{1}{2^{p+1}} \\
&= \frac{1}{2^p}
\end{aligned}$$

So $g \in V = B_{1/2^p}(f)$.

So V is open in S with respect to the product topology.

Suppose now that U is an open subbase element in S with respect to the product topology. Suppose U contains the point $a = <a(n)>_{n\in\mathbb{N}}$. It will suffice to find an ε such that $B_\varepsilon(a)$ (with respect to ρ) is contained in U.

Now U is of the form $\{x : f(n) \in W_n\}$ where $W_n = S_n$ for all n except, possibly, for some $k \in \mathbb{N}$ where W_k is open in S_k. Then there is an ε_1 such that

$$B_a = \{g \in S : \rho_k(a(k), g(k)) < \varepsilon_1\}$$

is a subset of U containing a. Then

$$\frac{\rho_k(a(k), g(k))}{2^k} < \frac{\varepsilon_1}{2^k}$$

Choose $\varepsilon = \varepsilon_1/2^k$. We then claim that $B_\varepsilon(a) \subseteq B_a \subseteq U$ (with respect to the metric, ρ).

Proof of claim: Let $g \in B_\varepsilon(a)$. Then

$$\rho(a,g) = \sum_{n\in\mathbb{N}} \frac{\rho_n(a(n),g(n))}{2^n} < \varepsilon = \frac{\varepsilon_1}{2^k}$$

Since

$$\frac{\rho_k(a(k), g(k))}{2^k} \leq \sum_{n\in\mathbb{N}} \frac{\rho_n(a(n), g(n))}{2_n} = \rho(a, g) < \varepsilon = \varepsilon_1/2^k$$

then this $g \in B_a$. So $B_\varepsilon(a) \subseteq B_a \subseteq U$, with respect to ρ. We conclude that U is open in (S, ρ).

Then, the metric space, (S, ρ), and the product space, (S, τ), have the same topology.

In the last part of this chapter we will be investigating how the product topology is involved in various examples (some of which can be quite intricate).

7.6 Example: The *Tychonoff plank*

We now briefly discuss the product of two ordinal spaces, for future reference. The ordinal space was introduced in definition 5.16.[4] In the following example, ω_1 represents the first uncountable ordinal, while, ω_0 represents the first countable infinite ordinal. Let W represent the ordinal space, $[0, \omega_1]$, and T represent the ordinal space, $[0, \omega_0]$. Recall (from the discussion which appears after the definition 5.16) that the elements of the subbase elements of the form $(\mu, \omega_1]$ and $[0, \beta)$.

Let

$$S = W \times T = [0, \omega_1] \times [0, \omega_0]$$

be the product space of the two given ordinal spaces. Then the elements of S can be viewed as ordered pairs, $(\alpha, \beta) \in W \times T$. Since both sets are linearly ordered, it doesn't hurt to visualize the product space, S, as a Cartesian plane of numbers where W represents the horizontal axis and T represents the vertical axis. We would then have $(0, 0)$ in the lower left corner and (ω_1, ω_0) in the top right corner. The topological space $S = [0, \omega_1] \times [0, \omega_0]$ equipped with this topology is commonly referred to by topologists as the

"Tychonoff plank"

It is worth keeping this in mind, for future reference. The subspace S^* of S defined as

$$S^* = S \backslash \{(\omega_1, \omega_0)\}$$

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[4] A more detailed study of the ordinals in the context of set theory is found in Set theory: An introduction to Axiomatic Reasoning, by Robert André

simply obtained by deleting the top right corner from the Tychonoff plank is appropriately referred to as the

"Deleted Tychonoff plank"

This is also standard nomenclature. As an open neighborhood base of the point, $(\beta, \mu) \in S$, we can use elements of the form

$$\mathscr{B}_{(\beta,\mu)} = \{(\alpha, \beta + 1) \times (\gamma, \mu + 1) : \alpha < \beta \text{ and } \gamma < \mu\}$$

The Tychonoff plank may appear to be a topological space that is, in many ways, similar to the product space $\mathbb{R} \times \mathbb{N}$ but, as we will eventually see, has quite different properties. Both the Tychonoff plank and the Deleted Tychonoff plank are useful topological spaces to remember.

7.7 Topic: Embedding a topological space in a product space.

In this section we will show how a particular family, $\mathscr{F}$, of continuous functions on a topological space, (S, τ), can be used to embed S in a product space whose factors are the range of the functions in $\mathscr{F}$. This method is exhibited in a theorem titled "The Embedding Theorem I". The *Embedding theorem I* applies only to topological spaces, S, in which singleton sets are closed. In order to understand this theorem one must be familiar with the following three very important notions in topology:

- Family of functions which *separates points and closed sets* of S,
- *Evaluation map* with respect to a family, $\mathscr{F}$, of functions.
- If $f : A \to T$ is a continuous function which maps (non-empty) A homeomorphically into T, we say that "f *embeds* A into T".

Definition 7.15 Let (S, τ_S) be a topological space and $\{(S_\alpha, \tau_\alpha) : \alpha \in \Gamma\}$ be an indexed family of non-empty topological spaces. Let $\mathscr{F} = \{f_\alpha : \alpha \in \Gamma\}$ be a set of *continuous functions*, $f_\alpha : S \to S_\alpha$, each one mapping S onto its range $f_\alpha[S] \subseteq S_\alpha$.

(a) We say that $\mathscr{F}$ *separates points and closed sets* if, whenever F is a closed subset of S and $x \notin F$, then there exists at least one function $f_\beta \in \mathscr{F}$ such that $f_\beta(x) \notin \text{cl}_{S_\beta} f_\beta[F]$.[5]

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[5] Similar expressions such as "the subsets A and B are completely separated" and "the real-valued function, $f : S \to \mathbb{R}$ separates A and B if $A \subseteq Z(f)$ and $B \subseteq Z(f - 1)$ for some $f \in C(X)$" will be defined when discussing normal spaces in definition 10.2.

(b) We define a function, $e_{\mathscr{F}} : S \to \prod_{\alpha\in\Gamma} S_\alpha$, as follows:

$$e_{\mathscr{F}}(x) = \ <f_\alpha(x)>_{\alpha\in\Gamma} \ \in \ \prod_{\alpha\in\Gamma} f_\alpha[S] \ \subseteq \ \prod_{\alpha\in\Gamma} S_\alpha$$

We refer to the function $e_{\mathscr{F}} : S \to \prod_{\alpha\in\Gamma} S_\alpha$ as the

"*evaluation map of S into $\prod_{\alpha\in\Gamma} S_\alpha$ with respect to $\mathscr{F}$*"

Theorem 7.16 *The embedding theorem I.* Let (S, τ_S) be a topological space in which every singleton set in S is a closed subset of S. Given an indexed family of non-empty topological spaces, $\{(S_\alpha, \tau_\alpha) : \alpha \in \Gamma\}$, let $\mathscr{F} = \{f_\alpha : \alpha \in \Gamma\}$ be a set of *continuous functions* where each f_α maps S onto its range, $f_\alpha[S]$, inside S_α.

(a) If $\mathscr{F}$ *separates points and closed sets* of its domain, S, and $\prod_{\alpha\in\Gamma} S_\alpha$ is equipped with the product topology, then the evaluation map,

$$e_{\mathscr{F}}(x) \ = < f_\alpha(x) >_{\alpha\in\Gamma} \ \in \ \prod_{\alpha\in\Gamma} f_\alpha[S] \ \subseteq \ \prod_{\alpha\in\Gamma} S_\alpha$$

with respect to $\mathscr{F}$, is both continuous and one-to-one on S.

(b) Furthermore, the function, $e_{\mathscr{F}} : S \to \prod_{\alpha\in\Gamma} S_\alpha$, maps S homeomorphically onto

$$e_{\mathscr{F}}[S] \ \subseteq \ \prod_{\alpha\in\Gamma} f_\alpha[S] \ \subseteq \ \prod_{\alpha\in\Gamma} S_\alpha$$

Hence this evaluation map embeds a homeomorphic copy of S into the product space, $\prod_{\alpha\in\Gamma} S_\alpha$.

Proof : We are given that (S, τ_S) is a topological space in which all singleton sets, $\{x\}$, are closed in S and a family of topological spaces, $\{S_\alpha : \alpha \in \Gamma\}$. For the set, $\mathscr{F} = \{f_\alpha : S \to S_\alpha\}_{\alpha\in\Gamma}$, of continuous functions on S, we define

$$e_{\mathscr{F}} : S \to \prod_{\alpha\in\Gamma} f_\alpha[S] \subseteq \prod_{\alpha\in\Gamma} S_\alpha$$

as an evaluation map with respect to $\mathscr{F}$.

(a) Note that, for each $\alpha \in \Gamma$ and $x \in S$,

$$(\pi_\alpha \circ e_{\mathscr{F}})(x) = \pi_\alpha(< f_\alpha(x) >_{\alpha\in\Gamma}) = f_\alpha(x)$$

Since, for each $\alpha \in \Gamma$, f_α is continuous, then so is $\pi_\alpha \circ e_{\mathscr{F}} : S \to f_\alpha[S]$. By Lemma 7.11, $e_{\mathscr{F}} : S \to \prod_{\alpha\in\Gamma} f_\alpha[S]$ is continuous.

We *claim* that $e_{\mathscr{F}} : S \to \prod_{\alpha\in\Gamma} f_\alpha[S]$ is one-to-one on S. Suppose a and b are distinct points in S. Then, since the single set $\{b\}$ is closed and $\mathscr{F}$ separates points and closed sets, there exists $\beta \in \Gamma$ such that $f_\beta(a) \notin \mathrm{cl}_{S_\beta} f_\beta[\{b\}]$. Then the β^{th} component of $e_{\mathscr{F}}(a) = <f_\alpha(a)>_{\alpha\in\Gamma}$ and $e_{\mathscr{F}}(b) = < f_\alpha(b) >_{\alpha\in\Gamma}$ are distinct and so $e_{\mathscr{F}}(a) \neq e_{\mathscr{F}}(b)$. We conclude that the evaluation map $e_{\mathscr{F}} : S \to \prod_{\alpha\in\Gamma} f_\alpha[S]$ is one-to-one on S, as *claimed.*

b) To prove that the evaluation map $e_{\mathscr{F}} : S \to \prod_{\alpha\in\Gamma} f_\alpha[S]$ embeds S into $\prod_{\alpha\in\Gamma} S_\alpha$, it will suffice to show that it is an open function and, then invoke Theorem 6.9.

Let U be a non-empty open subset of S with the point $u \in U$. Then $F = S \setminus U$ is closed in S. Since $\mathscr{F}$ separates points and closed sets, there exists $\beta \in \Gamma$ such that $f_\beta(u) \notin \mathrm{cl}_{S_\beta} f_\beta[F]$. That means, $f_\beta(u) \in S_\beta \setminus \left[\mathrm{cl}_{S_\beta} f_\beta[F]\right]$.

We now show that $e_{\mathscr{F}}[U]$ is open in $\prod_{\alpha\in\Gamma} S_\alpha$.

Note that,

$$\begin{aligned}(\pi_{\beta}\circ e_{\mathscr{F}})(u) &= \pi_\beta(e_{\mathscr{F}}(u)) \\ &= \pi_\beta(<f_\alpha(u)>_{\alpha\in\Gamma}) \\ &= f_\beta(u) \\ &\in S_\beta \setminus \left[\mathrm{cl}_{S_\beta} f_\beta[F]\right]\end{aligned}$$

Since $e_{\mathscr{F}}(u) \in \pi_\beta^{\leftarrow}\left[S_\beta \setminus \left[\mathrm{cl}_{S_\beta} f_\beta[F]\right]\right]$ and since π_β is continuous, then $\pi_\beta^{\leftarrow}\left[S_\beta \setminus \left[\mathrm{cl}_{S_\beta} f_\beta[F]\right]\right]$ is an open neighborhood of $e_{\mathscr{F}}(u)$ in $\prod_{\alpha\in\Gamma} S_\alpha$. It now suffices to show that

$$\pi_\beta^{\leftarrow}\left[S_\beta \setminus \left[\mathrm{cl}_{S_\beta} f_\beta[F]\right]\right] \subseteq e_{\mathscr{F}}[U]$$

Suppose $e_{\mathscr{F}}(a) \in \pi_\beta^{\leftarrow}\left[S_\beta \setminus \left[\mathrm{cl}_{S_\beta} f_\beta[F]\right]\right]$.

$$\begin{aligned}e_{\mathscr{F}}(a) \in \pi_\beta^{\leftarrow}\left[S_\beta \setminus \left[\mathrm{cl}_{S_\beta} f_\beta[F]\right]\right] &\Rightarrow (\pi_{\beta}\circ e_{\mathscr{F}})(a) \in \pi_\beta\left[\pi_\beta^{\leftarrow}\left[S_\beta \setminus \mathrm{cl}_{S_\beta} f_\beta[F]\right]\right] \\ &\Rightarrow f_\beta(a) \in \left[S_\beta \setminus \mathrm{cl}_{S_\beta} f_\beta[F]\,\right] \\ &\Rightarrow f_\beta(a) \notin \mathrm{cl}_{S_\beta} f_\beta[F] \\ &\Rightarrow a \in S \setminus F = S \setminus (S \setminus U) = U \\ &\Rightarrow e_{\mathscr{F}}(a) \in e[U]\end{aligned}$$

So $\pi_\beta^{\leftarrow}\left[S_\beta \setminus \left[\mathrm{cl}_{S_\beta} f_\beta[F]\right]\right]$ is an open neighborhood of $e_{\mathscr{F}}(u)$ which is entirely contained in $e_{\mathscr{F}}[U]$. We conclude $e_{\mathscr{F}}[U]$ is open in $\prod_{\alpha\in\Gamma} S_\alpha$ and so $e_{\mathscr{F}} : S \to \prod_{\alpha\in\Gamma} S_\alpha$ is a homeomorphism.

We have thus shown that $e_{\mathscr{F}} : S \to \prod_{\alpha\in\Gamma} S_\alpha$ embeds S into the product space, $\prod_{\alpha\in\Gamma} S_\alpha$.

7.8 Topic: The *Cantor set*: Viewed as the continuous image of a product space.

In this section we will discuss the Cantor set. We will provide a formal set-theoretic definition of the Cantor set and then study it from a topological point of view. It is often referred to because of its surprising set-theoretic and topological properties. We will study it in this chapter since its description extensively uses Cartesian products.

A prologue to a definition of the Cantor set.

Recall that $\mathbb{Z}^+ = \{1, 2, 3, \ldots, \}$. Let $D = \{0, 1, 2, \ldots, 8, 9\}$ (the ten digits from 0 to 9). So

$$\prod_{n\in\mathbb{Z}^+} D = D^{\mathbb{Z}^+}$$

represents the set of all functions mapping $\mathbb{Z}^+$ into D. The set, $\prod_{n\in\mathbb{Z}^+} D$ can be described as the set of all countably infinite ordered strings, of the form,

$$<m_n>_{n\in\mathbb{Z}^+}, \text{ where } m_n \in \{0, 1, 2, \ldots, 9\}$$

Suppose we define the function, $\varphi : \prod_{n\in\mathbb{Z}^+} D \to [0, 1]$ (the closed interval in $\mathbb{R}$ from 0 to 1) as follows:

$$\varphi(<m_1, m_2, m_3, \ldots, m_n, \ldots, >) = \sum_{n=1}^{\infty} \frac{m_n}{10^n}$$

Note that every number $x \in [0, 1]$ (represented in its infinite decimal expansion $0.m_1m_2m_3\cdots$) can be expressed in the form,

$$x = \sum_{n=1}^{\infty} \frac{m_n}{10^n} = 0.m_1m_2m_3\cdots$$

For example,

$$\varphi(<0, 0, 0, \ldots>) = \sum_{n=1}^{\infty} \frac{0}{10^n} = 0$$

$$\varphi(<9, 9, 9, \ldots>) = \sum_{n=1}^{\infty} \frac{9}{10^n} = 0.9999\ldots = 1$$

$$\varphi(<2, 9, 9, \ldots>) = 0.2999\ldots = 0.3000\ldots = \varphi(< 3, 0, 0, \ldots>)$$

and so φ maps $\prod_{n\in\mathbb{Z}^+} D$ *onto* [0,1] but is not necessarily one-to-one (since an endless string of 9's and an endless string of 0's may be mapped to the same element in [0, 1]). But the entire set [0,1] is, indeed, the image of $\prod_{n\in\mathbb{Z}^+} D$ under φ.

However, if we reduce the size of D to say, $D^* = \{0, 1, 2, 5, 7, 9\}$ (by removing the four digits 3, 4, 6, 8), then the function, $\varphi : \prod_{n\in\mathbb{Z}^+} D^* \to [0, 1]$, defined similarly, would produce a range which is, not surprisingly, a *proper* subset of $[0, 1]$ which would contain multiple gaps in it (since the digits 3, 4, 6 and 8 are lacking in the ordered strings of the domain).

If we set $D = \{0, 1, 2\}$ and every number x in $[0, 1]$ is expressed in its triadic expansion form[6], then the set

$$\varphi\left[\prod_{n\in\mathbb{Z}^+}\{0,1,2\}\right] = [0.000\ldots_3\,,\ 0.2222\ldots_3] = [0,1]$$

would look like,

$$\varphi(< m_n >_{n\in\mathbb{Z}^+}) = \sum_{n=1}^{\infty}\frac{m_n}{3^n} = 0.m_1m_2m_3\cdots_3 \in [0.000\ldots_3\,,\ 0.2222\ldots_3] = [0,1]$$

For example,

$$\begin{aligned}
\varphi(<0,0,0,\ldots>) = 0.0000\ldots_3 &= \sum_{n=1}^{\infty}\frac{0}{3^n}\\
\varphi(<2,2,2,\ldots>) = 0.2222\ldots_3 &= \sum_{n=1}^{\infty}\frac{2}{3^n} = 1\\
\varphi(<1,1,1,\ldots>) = 0.1111\ldots_3 &= \sum_{n=1}^{\infty}\frac{1}{3^n} = \frac{1}{2} \quad \text{(Check!)}\\
\varphi(<0,1,2,2,\ldots>) = 0.012222_3\ldots &= 0.0200000000\ldots_3 = \varphi(0,2,0,0,0,\ldots)
\end{aligned}$$

Note that, φ *is not one-to-one onto* $\prod_{n\in\mathbb{Z}^+}\{0, 1, 2\}$.

We reduce the set $\{0, 1, 2\}$ to $D = \{0, 2\}$ (by removing the digit 1), with $m_n \in \{0, 2\}$ and let $\varphi : \prod_{n\in\mathbb{Z}^+}\{0, 2\} \to [0, 1]$ be defined as,

$$\varphi(<m_n>_{n\in\mathbb{Z}^+}) = \sum_{n=1}^{\infty}\frac{m_n}{3^n} = 0.m_1m_2m_3\ldots_3 \in [0.000\ldots_3\,,\ 0.2222\ldots_3]$$

where $0.m_1m_2m_3\ldots_3$ contains only 0's and 2's. The set, $\varphi\left[\prod_{n\in\mathbb{Z}^+}\{0, 2\}\right]$ is now a *proper* subset of $[0.000\ldots_3\,,\ 0.2222\ldots_3] = [0, 1]$.

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[6] . . . , 0, 1, 2, 10, 11, 12, 100, 101, 102, 110, . . .

We will see that the "Cantor set" involves the particular function

$$\varphi : \prod_{n\in\mathbb{Z}^+}\{0,2\} \to [0,1]$$

who's range, $\varphi\left[\prod_{n\in\mathbb{Z}^+}\{0,2\}\right]$, is a proper subset of [0, 1].

We can now provide a formal set-theoretic definition of the Cantor set.

Definition 7.17 *The Cantor set: A set theoretic definition.* Suppose $\mathbb{Z}^+ = \mathbb{N}\backslash\{0\}$ and $\prod_{n\in\mathbb{Z}^+}\{0,2\} = \{0,2\}^{\mathbb{Z}^+}$, the set of all countably infinite sequences of 0's and 2's. Let the function

$$\varphi : \prod_{n\in\mathbb{Z}^+}\{0,2\} \to [0,1]$$

be defined as

$$\varphi(< m_n >_{n\in\mathbb{Z}^+}) = \sum_{n=1}^{\infty} \frac{m_n}{3^n}$$

where $\varphi(< m_n >_{n\in\mathbb{Z}^+}) = 0.m_1m_2m_3\ldots_3$ is an element of $[0,1]$ expressed in its triadic expansion form. Since the digit 1 is lacking in the ordered strings, φ is one-to-one on its domain and the range, $\varphi\,[\,\prod_{n\in\mathbb{Z}^+}\{0,2\}\,]$, is a *proper* subset of the interval, $[0.000\ldots_3\,,\, 0.2222\ldots_3] = [0,1]$, with multiple gaps in it. The one-to-one image,

$$\mathbf{C} = \varphi\ \left[\ \prod_{n\in\mathbb{Z}^+}\{0,2\}\ \right]$$

of the product, $\prod_{n\in\mathbb{Z}^+}\{0,2\}$, under φ, is a proper subset of $[0.000\ldots_3\,,\, 0.2222\ldots_3] = [0,1]$. The set, $\mathbf{C}$, is referred to as the *Cantor set*.

Then the Cantor set can be seen as

$$\mathbf{C} = \left\{\sum_{n=1}^{\infty} \frac{m_n}{3^n} : m_n \in \{0,2\}\right\}$$

all possible values of the infinite sum $\sum_{n=1}^{\infty} \frac{m_n}{3^n}$ where $m_n = 0$ or 2.

What does the Cantor set look like, as a subset of $[0,1]$*?*

The definition of the "Cantor Set", $\mathbf{C}$, states that $\mathbf{C}$ is a particular subset of the closed unit interval, $[0,1]$. This set is the range,

$\varphi\,[\,\prod_{n\in\mathbb{Z}^+}\{0,2\}\,]$, under the well-defined function, φ, with domain, $\prod_{n\in\mathbb{Z}^+}\{0,2\}$. This definition, by itself, is not very useful when trying to visualize what kind of subset of $[0,1]$ it represents. If the reader is willing to go through the trouble of verifying this, $\mathbf{C}$ does not contain any of the points in the open intervals, $(1/3, 2/3)$, $(1/3^2, 2/3^2)$, $(7/3^2, 8/3^2)$. Since φ maps $\prod_{n\in\mathbb{Z}^+}\{0,1,2\}$ onto $[0,1]$, then φ will map $\prod_{n\in\mathbb{Z}^+}\{0,2\}$ only onto a proper subset of [0, 1]. Such gaps will occur. It doesn't take long before one sees a pattern emerge. We normally view the Cantor set as a set which is inductively constructed in stages by successively defining a nested sequence, $C_0, C_1, C_2, \ldots$, of subsets of [0,1]. Ultimately we define the Cantor set as being the intersection, $\mathbf{C} = \cap_{n\in\mathbb{N}} C_n$, of all of these (after convincing ourselves that $\mathbf{C}$ would not be empty). We can visualize this procedure as follows.

$$\begin{array}{l}
C_0 = [0,1] \\
C_1 = C_0 \setminus (1,3) \\
C_2 = C_1 \setminus (\frac{1}{3^2}\frac{2}{3^2}) \cup (\frac{7}{3^2}\frac{8}{3^2}) \\
C_3 = C_2 \setminus \text{...open middle thirds in } C_2 \\
\quad\vdots \\
C_{n+1} = C_n \setminus \text{...open middle thirds of the remaining intervals in } C_n \\
\quad\vdots
\end{array}$$

The construction of C_n is normally described by saying " … to obtain C_n, subtract open middle thirds from C_{n-1}" After inductively obtaining an infinite sequence of nested sets, $\{C_n\}_{n\in\mathbb{N}}$, in this way we define the Cantor set as being the infinite intersection

$$\mathbf{C} = \cap_{n=0}^{\infty} C_n$$

We visualize the Cantor set geometrically (up to the construction of C_5) as follows.

Note that, in the definition 7.17 of the Cantor set $\mathbf{C} = \varphi[\prod_{n\in\mathbb{Z}^+}\{0,2\}] \to [0,1]$ where

$$\varphi(< m_n >_{n\in\mathbb{Z}^+}) = \sum_{n=1}^{\infty} \frac{m_n}{3^n},$$

the topology of the set $\mathbf{C}$ involved was not discussed. So we didn't speak of the "continuity" of φ on the product $\prod_{n\in\mathbb{Z}^+}\{0,2\}$. We will define topologies of all sets involved in the most natural way. We will equip the set $\{0,2\}$ with the discrete topology, the set $\prod_{n\in\mathbb{Z}^+}\{0,2\}$ with the product topology, and finally, the Cantor set, $\mathbf{C}$, with the subspace topology inherited from $\mathbb{R}$. We will now show that, the Cantor set, $\mathbf{C}$, is a homeomorphic copy of the product space, $\prod_{n\in\mathbb{Z}^+}\{0,2\}$.

Theorem 7.18 The one-to-one function, $\varphi : \prod_{n\in\mathbb{Z}^+}\{0,2\} \to [0,1]$, defined as,

$$\varphi(< m_n >_{n\in\mathbb{Z}^+}) = \sum_{n=1}^{\infty} \frac{m_n}{3^n}$$

maps the product space $\prod_{n\in\mathbb{Z}^+}\{0,2\}$ homeomorphically onto the Cantor set,

$$\mathbf{C} = \varphi\left[\prod_{n\in\mathbb{Z}^+}\{0,2\}\right]$$

a subset of $[0,1]$. So the Cantor set is a topological copy of the product space, $\prod_{n\in\mathbb{Z}^+}\{0,2\}$, contained in $[0,1]$.

Proof: Given the one-to-one function $\varphi : \prod_{n\in\mathbb{Z}^+}\{0,2\} \to [0,1]$ where

$$\varphi(< m_n >_{n\in\mathbb{Z}^+}) = \sum_{n=1}^{\infty} \frac{m_n}{3^n}$$

Let $\varepsilon > 0$. First note that, for all $< m_n >_{n\in\mathbb{Z}^+} \in \prod_{n\in\mathbb{Z}^+}\{0,2\}$,

$$|\varphi(< m_n >_{n\in\mathbb{Z}^+})| \leq \sum_{n=1}^{\infty} \frac{2}{3^n} = 1$$

Then the sequence,

$$\left\{ \sum_{n=1}^{\infty} \frac{2}{3^n} - \sum_{n=1}^{k} \frac{2}{3^n} \right\}_{k\in\mathbb{Z}^+}$$

converges to zero as $k \to \infty$. So there exists N such that

$$\sum_{n=N+1}^{\infty} \frac{2}{3^n} < \varepsilon \qquad (*)$$

Claim #1. The function φ is continuous. If $< k_n >_{n\in\mathbb{Z}^+} \in \prod_{n\in\mathbb{Z}^+}\{0,2\}$ then

$$\varphi(< k_n >_{n\in\mathbb{Z}^+}) = \sum_{n=1}^{\infty}\frac{k_n}{3^n} = x \in \mathbf{C}$$

where $< k_n >$ is a string of 0s and 2s.

Let $B_\varepsilon(x)$ be and open interval in $\mathbb{R}$ with center x and radius ε. Then $B_\varepsilon(x) \cap \mathbf{C}$ is an open neighborhood of x in $\mathbf{C}$. To show continuity of φ at $< k_n >_{n\in\mathbb{Z}^+}$, it will suffice to find an open neighborhood, U, of $< k_n >_{n\in\mathbb{Z}^+}$ such that $\varphi[U] \subseteq B_\varepsilon(x)\cap\mathbf{C}$.

Since the series, $\sum_{n=1}^{\infty}\frac{k_n}{3^n}$, converges to x, then the sequence $\left\{x - \sum_{n=1}^{m}\frac{k_n}{3^n}\right\} = \left\{\sum_{n=m+1}^{\infty}\frac{k_n}{3^n}\right\}$ converges to zero as m tends to infinity. Then,

$$\left|\sum_{n=N+1}^{\infty}\frac{k_n}{3^n}\right| \leq \sum_{n=N+1}^{\infty}\frac{2}{3^n} < \varepsilon \qquad \text{(By *)}$$

Let $U = \pi_1^{\leftarrow}[\{k_1\}] \cap \cdots \cap \pi_N^{\leftarrow}[\{k_N\}]$ (wlog) be an open base neighborhood of $< k_n >$ in $\prod_{n\in\mathbb{Z}^+}\{0,2\}$ (Each $\{k_i\}$ is an open subset of $\{0,2\}$). Suppose $< b_n >_{n\in\mathbb{Z}^+}$ is some element in U. This implies $b_1 = k_1, b_2 = k_2, \ldots, b_N = k_N$. So

$$\begin{aligned}
|x - \varphi(< b_n >_{n\in\mathbb{Z}^+})| &= \left|x - \sum_{n=1}^{\infty}\frac{b_n}{3^n}\right| \\
&= \left|\sum_{n=1}^{\infty}\frac{k_n}{3^n} - \sum_{n=1}^{\infty}\frac{b_n}{3^n}\right| \\
&\leq 0 + \sum_{n=N+1}^{\infty}\frac{|k_n - b_n|}{3^n} \\
&\leq \sum_{n=N+1}^{\infty}\frac{2}{3^n} \\
&< \varepsilon
\end{aligned}$$

Then $\varphi[U] \subseteq B_\varepsilon(x)\cap\mathbf{C}$. Then φ is continuous at $< k_n >_{n\in\mathbb{Z}^+}$, and so at all points of $\prod_{n\in\mathbb{Z}^+}\{0,2\}$. Then φ is continuous on $\prod_{n\in\mathbb{Z}^+}\{0,2\}$, as claimed.

Claim #2. The function φ is homeomorphic. To prove that φ is a homeomorphism it will suffice to show it is an open function.
For $y = < k_n >_{n\in\mathbb{Z}^+} \in \prod_{n\in\mathbb{Z}^+}\{0,2\}$, $\sum_{n=1}^{\infty}\frac{k_n}{3^n}$ converges to $\varphi(y)$.

Let $U = \pi_{j_1}^{\leftarrow}[\{k_{j_1}\}] \cap \cdots \cap \pi_{j_m}^{\leftarrow}[\{k_{j_m}\}]$ be an arbitrary open neighborhood base element of $< k_n >_{n\in\mathbb{Z}^+}$.
To show that φ is open, it will suffice to find some ε such that $B_\varepsilon(\varphi(y))\cap \mathbf{C} \subseteq \varphi[U]$.
Let $N = \max\{j_1, j_2, \dots, j_m\}$ and let $\varepsilon = \frac{1}{3^{N+1}}$. (**)

We claim that, if $|\,\varphi(< k_n >_{n\in\mathbb{Z}^+}) - \varphi(< z_n >_{n\in\mathbb{Z}^+})\,| < \varepsilon$, then $\varphi(< z_n >_{n\in\mathbb{Z}^+}) \in \varphi[U]$.

$$\begin{aligned}
|\,\varphi(< k_n >_{n\in\mathbb{Z}^+}) - \varphi(< z_n >_{n\in\mathbb{Z}^+})\,| &= \left|\sum_{n=1}^{\infty}\frac{k_n}{3^n} - \sum_{n=1}^{\infty}\frac{z_n}{3^n}\right| \\
&= \left|\sum_{n=1}^{\infty}\frac{k_n - z_n}{3^n}\right| \\
&= \left|\sum_{n=1}^{N}\frac{k_n - z_n}{3^n} + \sum_{n=N+1}^{\infty}\frac{k_n - z_n}{3^n}\right| \\
&< \varepsilon
\end{aligned}$$

Then

$$-\varepsilon = \frac{-1}{3^{N+1}} < \sum_{n=1}^{N}\frac{k_n - z_n}{3^n} + \sum_{n=N+1}^{\infty}\frac{k_n - z_n}{3^n} < \frac{1}{3^{N+1}} = \varepsilon$$

$$\frac{-1}{3^{N+1}} - \sum_{n=N+1}^{\infty}\frac{k_n - z_n}{3^n} < \sum_{n=1}^{N}\frac{k_n - z_n}{3^n} < \frac{1}{3^{N+1}} - \sum_{n=N+1}^{\infty}\frac{k_n - z_n}{3^n}$$

After some computation we obtain,

$$\frac{-4}{3^{N+1}} < \sum_{n=1}^{N}\frac{k_n - z_n}{3^n} < \frac{4}{3^{N+1}} \quad \text{(Verify this!)}$$

To show that $\varphi(< z_n >_{n\in\mathbb{Z}^+}) \in \varphi[U]$, it suffices to show that, $y_n - z_n = 0$ for $n = 1$ to N.
For $m = 1$ to N, let

$$S_m = \sum_{n=1}^{m}\frac{k_n - z_n}{3^n}$$

Suppose $k_1 - z_1 = 2$ or -2; then $S_1 = \pm 2/3 \notin (-\frac{4}{3^{N+1}}, \frac{4}{3^{N+1}})$. So $k_1 - z_1 = 0$.

Suppose $k_m - z_m = 0$ for $m < N$ and $k_{m+1} - z_{m+1} = 2$ or -2. Then $S_{m+1} = \pm 2/3 \notin (-\frac{4}{3^{N+1}}, \frac{4}{3^{N+1}})$. So $k_{m+1} - z_{m+1} = 0$.

We then have $y_n - z_n = 0$ for $n = 1$ to N.

This is precisely what was needed to show that $\varphi(\{z_n\}) \in \varphi[U]$. This means that, for our choice of $\varepsilon = \frac{1}{3^{N+1}}$ (at (**)), $B_\varepsilon(\varphi(\{k_n\}) \cap \mathbf{C} \subseteq \varphi[U]$. So φ is an open map on $\prod_{n\in\mathbb{Z}^+}\{0,2\}$. So φ is a homeomorphism, as claimed.

The theorem is proved, as required

The Cantor set was defined as being the range $\varphi[\,\prod_{n\in\mathbb{Z}^+}\{0,2\}\,]$ of $\prod_{n\in\mathbb{Z}^+}\{0,2\}$. But the above statement shows much more. Since we have shown $\prod_{n\in\mathbb{Z}^+}\{0,2\}$ is a homeomorphic copy of $\mathbf{C}$, then, topologically speaking, the subspace, $\mathbf{C}$, of $[0,1]$ we call the Cantor set "is" the product space, $\prod_{n\in\mathbb{Z}^+}\{0,2\}$.

> *A topological definition of the Cantor set:* The Cantor set is the product space,
>
> $$\prod_{n\in\mathbb{Z}^+}\{0,2\} = \{0,2\}^{\mathbb{Z}^+}$$
>
> where $\{0,2\}$ is the two-point discrete space and $\{0,2\}^{\mathbb{Z}^+}$ is equipped with the product topology.[7]

The topological point of view provides a different perspective on the nature of $\mathbf{C}$. Since the Cantor set is homeomorphic to a subset of the metrizable space $[0,1]$ then the Cantor set is a metrizable space.

About the cardinality of $\mathbf{C}$.

Our geometric description of the construction of the Cantor set, $\mathbf{C}$, showed that $\mathbf{C}$ was obtained by successively removing open middle-third interval from a previous set, leaving behind, at least, the endpoints of countably many closed intervals. The endpoints, all of them of the form, $\frac{m_n}{3_n}$, $(m_n \in \{0,2\})$ are rationals and are never removed and so must belong to $\mathbf{C}$. This may lead one to believe that $\mathbf{C}$ is

---

[7] For curious interested readers, there are other topological characterizations such as *"the Cantor set is homeomorphic to those spaces which are compact, metrizable, totally disconnected and perfect (a space is said to be perfect if none of its points are open)"*.

countably infinite. But our formal definition of **C** (as well as the previous theorem) shows that this cannot be so. It states that **C** can be mapped homeomorphically onto the uncountable set $\prod_{n\in\mathbb{Z}^+}\{0,2\}$. Then, $|\mathbf{C}| = 2^{|Z^+|} = 2^{\aleph_0} = c = |\mathbb{R}|$. So **C** is uncountable. We leave the reader to reflect on the intriguing question: If x is a point in **C** which is not an endpoint of a middle third, what does it look like? How can it be that a "non-endpoint" is left behind?

Is the Cantor topological space discrete?

By this question, we are wondering whether every single point of **C** is both open and closed. We have shown that the Cantor set, **C**, is the homeomorphic image of the product space, $\prod_{n\in\mathbb{Z}^+}\{0,2\}$. If every point of **C** was open, then every point of $\prod_{n\in\mathbb{Z}^+}\{0,2\}$ would be open. But this is easily verified not to be the case, since no point of the product space can contain a basic open set of the product space, $\prod_{n\in\mathbb{Z}^+}\{0,2\}$. Those spaces in which none of its singleton sets are open are referred to as being *perfect* spaces. So the Cantor set is a perfect space.

So **C** can be viewed as a sprinkling of an uncountable set of points in $[0,1]$. It is interesting to note that **C** can continuously be mapped onto $[0,1]$ as the following theorem shows.

Theorem 7.19 There is a continuous function, $\delta : \mathbf{C} \to [0,1]$, which maps the Cantor set, **C**, onto the closed interval $[0,1]$.

Proof: Recall that the function, $\varphi : \prod_{n\in\mathbb{Z}^+}\{0,2\} \to \mathbf{C}$, maps $\prod_{n\in\mathbb{Z}^+}\{0,2\}$ homeomorphically onto **C**. It is defined as

$$\varphi(< m_n >_{n\in\mathbb{Z}^+}) = \sum_{n=1}^{\infty} \frac{m_n}{3^n}$$

We begin by defining a similar function, $\psi : \prod_{n\in\mathbb{Z}^+}\{0,2\} \to [0,1]$ which maps the same set, $\prod_{n\in\mathbb{Z}^+}\{0,2\}$, onto [0,1]. It is defined as,

$$\psi(< m_n >_{n\in\mathbb{Z}^+}) = \sum_{n=1}^{\infty} \frac{m_n}{2^{n+1}}$$

We claim that ψ is continuous on its domain and onto the closed interval $[0,1]$. To see that ψ is onto $[0,1]$ simply note that the expression

$$\sum_{n=1}^{\infty} \frac{m_n}{2^{n+1}} = \sum_{n=1}^{\infty} \frac{m_n/2}{2^n} = \sum_{n=1}^{\infty} \frac{s_n}{2^n} = 0.s_1s_2s_3s_4\ldots_2 \in [0,1]$$

where each s_n is 0 or 1 and $0.s_1s_2s_3s_4\ldots_2$ is simply the dyadic expansion for an element in $[0,1]$.[8]

To prove continuity of ψ proceed precisely as for φ in the previous theorem.

We define $\delta : \mathbf{C} \to [0,1]$ as

$$\delta = \psi{\circ}\varphi^{\leftarrow}$$

where $\varphi^{\leftarrow}$ continuously maps $\mathbf{C}$ one-to-one and onto $\prod_{n\in\mathbb{Z}^+}\{0,2\}$ and ψ continuously maps $\prod_{n\in\mathbb{Z}^+}\{0,2\}$ onto $[0,1]$.[9] So δ continuously maps $\mathbf{C}$ onto $[0,1]$, as required.

7.9 Topic: Product spaces are "commutative".

What effect does it have on the topology of a product space if we alter the order of its factors? We investigate this question by first considering a particular case in the following example.

Example 8. Show that for topological spaces S_1, S_2 and S_3,

$$S = S_1 \times S_2 \times S_3 \text{ and } T = S_3 \times S_1 \times S_2$$

are homeomorphic.

Solution: Let $S = S_1 \times S_2 \times S_3$ and $T = S_3 \times S_1 \times S_2$. Let $q : \{1,2,3\} \to \{3,1,2\}$ be the one-to-one function where $q(1) = 3$, $q(2) = 1$, $q(3) = 2$. So $T = S_{q(1)} \times S_{q(2)} \times S_{q(3)} = S_3 \times S_1 \times S_2$.

Let $q^* : S \to T$ be defined as follows:

$$q^*[S_1 \times S_2 \times S_3] = S_{q(1)} \times S_{q(2)} \times S_{q(3)} = S_3 \times S_1 \times S_2$$

If $(a,b,c) \in S$ then

$$q^*(a,b,c) = (c,a,b) \in T$$

---

[8] For example, if $y =< m_n >_{n\in\mathbb{Z}^+}$ is the string $\{2,0,2,0,2,0,\ldots,\}$, then $\psi(y) = 0.10101010\ldots$ a point in [0, 1] in dyadic expansion form. The function ψ is easily seen to be onto $[0,1]$. For example, given $x = [0.001001001\ldots,]$, $\psi(\{0,0,2,0,0,2,\ldots\}) = x$.

[9] Note that ψ is not one-to-one since ψ maps the two distinct strings $\{0,0,2,2,2.\ldots,\}$ and $\{0,1,0,0,0.\ldots,\}$ to the same point $0.001111\ldots = 0.01000000\ldots$

Claim: We claim that $q^* : S_1 \times S_2 \times S_3 \to S_{q(1)} \times S_{q(2)} \times S_{q(3)}$ is a homeomorphism.

Proof of claim: We will first show that the three functions

$$\begin{array}{rcl} \pi_{q(3)} \circ q^* : S_1 \times S_2 \times S_3 & \to & S_{q(3)} \\ \pi_{q(2)} \circ q^* : S_1 \times S_2 \times S_3 & \to & S_{q(2)} \\ \pi_{q(1)} \circ q^* : S_1 \times S_2 \times S_3 & \to & S_{q(1)} \end{array}$$

are continuous. If so we can apply Lemma 7.11, to conclude q^* is continuous.

Let U be an open subset of $S_{q(3)} = S_2$.

$$\begin{array}{rcl} (\pi_{q(3)} \circ q^*)^{\leftarrow}[U] & = & q^{*\leftarrow}[\pi_{q(3)}{}^{\leftarrow}[U]] \\ & = & q^{*\leftarrow}[S_{q(1)} \times S_{q(2)} \times U] \\ & = & S_1 \times S_2 \times U \end{array}$$

So $\pi_{q(3)} \circ q^*$ is continuous. We mimic these steps to show that both $\pi_{q(2)} \circ q^*$ and $\pi_{q(1)} \circ q^*$ are continuous.

By Lemma 7.11, $q^* : S \to T$ is continuous.

By a similar reasoning, $q^{*\leftarrow} : T \to S$ is continuous.

We conclude that $q^* : S_1 \times S_2 \times S_3 \to S_{q(1)} \times S_{q(2)} \times S_{q(3)}$ is a homeomorphism, as required.

In the next theorem we generalize the proof in the example to one involving larger product spaces.

Theorem 7.20 Let $S = \prod_{\alpha \in I} S_\alpha$ and let $q : I \to J$ be a one-to-one function mapping I onto an indexing set, J. Then there is a homeomorphism q^* mapping S onto $T = \prod_{\alpha \in I} S_{q(\alpha)}$.

Proof THe proof follows the pattern illustrated in the example, and so is omitted.

The above theorem confirms that “altering the order of the factors of a product space produces another product space which is homeomorphic to the original one”.

7.10 Topic: Product spaces are “associative”.

We address the question: How can we simplify the product of a product spaces? We begin by illustrating what we mean by this question.

Let $I = \{1, 2, \ldots, 12\}$, $A = \{1, 2, 3, 4\}$, $B = \{5, 6\}$, and $C = \{7, 8, \ldots, 12\}$. The set I is the disjoint union the of sets A, B and C. Let $\{(S_\alpha, \tau_\alpha) : \alpha \in \{1, 2, \ldots, 12\}$ be a set of twelve topological spaces. Consider the two product spaces S and Y defined as follows:

$$S = \textstyle\prod_{\alpha \in I} S_\alpha \quad \text{and} \quad Y = \prod_{\alpha \in A} S_\alpha \times \prod_{\alpha \in B} S_\alpha \times \prod_{\alpha \in C} S_\alpha$$

We wonder, “How do the two product spaces compare?”. We see that S is a product space with twelve factors, while Y is a product space of only three factors. This observation is sufficient to conclude that $S \neq Y$. On the other hand, we see that each of the three factors of Y are themselves product spaces, all three with factors, S_α, identical to the ones found in S. While we have excluded, the possibility that S equals Y, it would be reasonable to suspect that these two spaces are homeomorphic copies of each other.

Example 9. Let $\{S_1, S_2, \ldots, S_6\}$ be a set of non-empty topological spaces. Suppose $I = \{1, 2, \ldots, 6\}$ is the union of three disjoint non-empty subsets, $A = \{1, 2\}$, $B = \{3, 4, 5\}$ and $C = \{6\}$ where the elements in $A \cup B \cup C$ respect the order in which they appear in I. Show that

$$S = \textstyle\prod_{i \in I} S_i \quad \text{and} \quad T = [S_1 \times S_2] \times [S_3 \times S_4 \times S_5] \times S_6$$

are homeomorphic product spaces.

Solution : We define three functions,

$$\begin{aligned} f_A : S &\to S_1 \times S_2 \\ f_B : S &\to S_3 \times S_4 \times S_5 \\ f_C : S &\to S_6 \end{aligned}$$

as

$$\begin{array}{rcl} f_A(x_1,x_2,x_3,x_4,x_5,x_6) & = & (x_1,x_2) \\ f_B(x_1,x_2,x_3,x_4,x_5,x_6) & = & (x_3,x_4,x_5) \\ f_C(x_1,x_2,x_3,x_4,x_5,x_6) & = & x_6 \end{array}$$

Note that,

$$\begin{array}{rcl} \text{for } j\in\{1,2\}, (\pi_j{\circ}f_A)(<x_i>_{i\in I\}}) & = & \pi_j(x_1,x_2)=x_j\in\{x_1,x_2\} \\ \text{for } j\in\{3,4,5\}, (\pi_j{\circ}f_B)(<x_i>_{i\in i\in I}) & = & \pi_j(x_3,x_4,x_5)=x_j\in\{x_3,x_4,x_5\} \\ \text{for } j\in\{6\}, (\pi_j{\circ}f_C)(<x_i>_{i\in i\in I}) & = & \pi_j(x_6)=x_j\in\{x_6\} \end{array}$$

It is easily verified that the six functions, $\pi_1{\circ}f_A$, $\pi_2{\circ}f_A$ $\pi_3{\circ}f_B$, $\pi_4{\circ}f_B$, $\pi_5{\circ}f_B$ and $\pi_6{\circ}f_C$ pull-back basic open sets in S_j to basic open sets in S and so are continuous on S.

By Lemma 7.11, the functions f_A, f_B, and f_C are all continuous on S.

We now consider the function

$$f:\prod_{i\in I}S_i\to[S_1\times S_2]\times[S_3\times S_4\times S_5]\times S_6$$

defined as

$$\begin{array}{rcl} f(<x_i>_{i\in I}) & = & (\ f_A(<x_i>_{i\in I}),\ f_B(<x_i>_{i\in I}),\ f_C(<x_i>_{i\in I})\ ) \\ & = & (\ (x_1,x_2),(x_3,x_4,x_5),x_6\ ) \\ & \in & [S_1\times S_2]\times[S_3\times S_4\times S_5]\times S_6 \end{array}$$

Claim #1. We claim that f is one-to-one and continuous on $\prod_{i\in I}S_i$.

Since f_A, f_B and f_C are all three continuous then, by theorem 7.12, $f:S\to T$ is continuous on its domain S. Also, since I is the union of disjoint A, B and C, then f maps S one-to-one and onto T. So f is one-to-one and continuous on S, as claimed.

Claim #2. We claim that f is a homeomorphism.

To show that f is a homeomorphism it suffices to show that f is an open map.

Let $V=\pi_1^{\leftarrow}[U_1]\cap\pi_2^{\leftarrow}[U_2]\cap\cdots\cap\pi_6^{\leftarrow}[U_6]$ be a basic open set in $S=\prod_{i\in I}S_i$. Then

$$f[V]\ =\ f_A[V]\times f_B[V]\times f_C[V]\ \subseteq\ [S_1\times S_2]\times[S_3\times S_4\times S_5]\times S_6$$

Then

$$\begin{aligned} f_A[V] &= f_A[\pi_1^{\leftarrow}[U_1] \cap \pi_2^{\leftarrow}[U_2] \cap \cdots \cap \pi_6^{\leftarrow}[U_6]] \\ &= f_A[U_1 \times U_2 \times \cdots U_6] \\ &= U_1 \times U_2 \end{aligned}$$

$$\begin{aligned} f_B[V] &= U_3 \times U_4 \times U_5 \\ f_C[V] &= U_6 \end{aligned}$$

So $f_A[V]$, $f_B[V]$ and $f_C[V]$ are open in $[S_1 \times S_2]$, $[S_3 \times S_4 \times S_5]$ and S_6. respectively.

Then $f[V]$ is open in $[S_1 \times S_2] \times [S_3 \times S_4 \times S_5] \times S_6$.

We conclude that

$$f : S_1 \times S_2 \times S_3 \times S_4 \times S_5 \times S_6 \quad \rightarrow \quad [S_1 \times S_2] \times [S_3 \times S_4 \times S_5] \times S_6$$

is a homeomorphism. So S and $[S_1 \times S_2] \times [S_3 \times S_4 \times S_5] \times S_6$ are homeomorphic sets, as required.

In the next theorem we generalize the proof in the example to one involving larger product spaces.

Theorem 7.21 Let $\{(S_\alpha, \tau_\alpha) : \alpha \in I\}$ be a set of non-empty topological spaces. Suppose I is the disjoint union of three non-empty subsets, A, B and C where the elements in $A \cup B \cup C$ respect the order in which they appear in I. Then

$$S = \prod_{\alpha \in I} S_\alpha \quad \text{and} \quad Y = \prod_{\alpha \in A} S_\alpha \times \prod_{\alpha \in B} S_\alpha \times \prod_{\alpha \in C} S_\alpha$$

are homeomorphic product spaces.

Proof : The proof mimics the method illustrated in the example and so is omitted.

Example 10. In an example found on page 111 we defined the space

$$(\psi, \tau)$$

where $\psi = \mathbb{N} \cup \mathscr{M}$ where $\mathscr{M}$ is a collection of infinite subsets of $\mathbb{N}$ in which every pair has finite intersection and is such that $\mathscr{M}$ is maximal with respect to this property. Show that $\mathscr{M}$ is uncountably large set.

Solution: In that example we inductively constructed an infinite family

$$\mathscr{D} = \cup\{\mathscr{D}_n : n \in \mathbb{N}\}$$

of distinct subsets of $\mathbb{N}$ where

$$\begin{array}{rcl} \mathscr{D}_0 & = & \{A_{(0,0)}\} \\ \mathscr{D}_1 & = & \{A_{(1,1)}, A_{(1,2)}\} \\ \mathscr{D}_2 & = & \{A_{(2,1)}, A_{(2,2)}, A_{(2,3)}, A_{(2,2^2)}\} \\ & \vdots & \\ \mathscr{D}_n & = & \{A_{(n,k)} : k = 1 \text{ to } 2^n\} \\ & \vdots & \end{array}$$

and

$$D_n = \{x_{(n,k)} : k = 1 \text{ to } 2^n\}$$

are distinct points such that $x_{(n,k)} \in A_{(n,k)} \backslash D_{n-1}$.

$$\begin{array}{rcl} D_0 & = & \{x_{(0,0)}\} \\ D_1 & = & \{x_{(1,1)}, x_{(1,2)}\} \\ D_2 & = & \{x_{(2,1)}, x_{(2,2)}, x_{(2,3)}, x_{(2,2^2)}\} \\ & \vdots & \\ D_n & = & \{x_{(n,k)} : k = 1 \text{ to } 2^n\} \\ & \vdots & \end{array}$$

Consider the map $\varphi : \prod_{i=1}^{\infty}\{0,1\}_i \to [0,1]_2$ defined as

$$\varphi(m_1, m_2, m_3, \ldots,) = 0.m_1m_2m_3\cdots_2 = \sum_{i=1}^{\infty} \frac{m_i}{2^i}$$

If we exclude those elements of $\prod_{i=1}^{\infty}\{0,1\}_i$ which end in an infinite sequence of 1s, the function is one-to-one and onto $[0,1]_2$ a set of cardinality $2^{\aleph_0}$. We can index the elements of D_n as follows:

$$\begin{aligned}
D_0 &= \{x_0\} \\
D_1 &= \{x_{(1,\,0.0)}, x_{(1,\,0.1)}\} \\
D_2 &= \{x_{(2,\,0.00)}, x_{(2,\,0.01)}, x_{(2,\,0.10)}, x_{(2,\,0.11)}\} \\
D_3 &= \{x_{(3,\,0.000)}, x_{(3,\,0.001)}, x_{(3,\,0.010)}, x_{(3,\,0.011)}, \ldots, x_{(3,\,0.111)}\} \\
&\vdots \\
D_n &= \{x_{(n,k)} : k = 1 \text{ to } 2^n\} \\
&\vdots
\end{aligned}$$

This gives rise to a an infinite family of sets $\mathscr{B} = \{B_q : q \in J\}$ where the sequence $B_k = \{x_{(n,\,q)} : n \in \mathbb{N}\}$ converges to a point in $[0,1]_2$.

So the cardinality of J is $2^{\aleph_0}$. If $s \neq t$, then $B_s \cap B_t$ is finite.

So the cardinality of $\mathscr{M}$ is larger than $\aleph_0$.

Concepts review.

1. Give a general definition of a Cartesian product of sets.
2. Provide two definitions of the Cantor set.
3. Define the *product topology* on a Cartesian product.
4. Define the *box topology* on a Cartesian product.
5. Define what we mean by *product space.*
6. What effect does changing the order of the factors have on the topology of a product space.
7. Describe the *Tychonoff plank* and its topology.
8. What does it mean to say that a set of functions $\mathscr{F}$ separates points and closed sets of a set S.
9. Define evaluation map with respect to a set of functions $\mathscr{F}$.
10. State the *Embedding theorem.*

11. The embedding theorem describes a homeomorphism between a topological space and a product space. What are these spaces? What is the homeomorphism?
12. The Cantor set equipped with the usual topology is shown to be homeomorphic to a product space. Which one? Describe the homeomorphism.
13. Describe a continuous function which maps the Cantor set, C, continuously onto the closed interval $[0,1]$.
14. Is the Cantor set a countable subset of $\mathbb{R}$?
15. Is the Cantor set, equipped with the subspace topology, discrete? Is it second countable?

EXERCISES

1. Show that $\prod_{\alpha\in\Gamma} S_\alpha$ is dense in $\prod_{\alpha\in\Gamma} T_\alpha$ if and only if, for each $\alpha \in \Gamma$, S_α is dense in T_α.
2. Suppose that, for each $\alpha \in \Gamma$, $S_\alpha \subseteq T_\alpha$. Then $\prod_{\alpha\in\Gamma} S_\alpha$ is a subset of the product space, $\prod_{\alpha\in\Gamma} T_\alpha$. Show that the product topology on $\prod_{\alpha\in\Gamma} S_\alpha$ is the same as the subspace topology $\prod_{\alpha\in\Gamma} S_\alpha$ inherits from $\prod_{\alpha\in\Gamma} T_\alpha$.
3. Given the product space $\prod_{\alpha\in\Gamma} S_\alpha$, show that the projection map $\pi_\alpha : \prod_{\alpha\in\Gamma} S_\alpha \to S_\alpha$ is an open function. Is it a closed function?
4. If in the product space, $S = \prod_{\alpha\in\Gamma} S_\alpha$, each S_α is discrete, describe the open subsets of S.
5. If in the product space, $S = \prod_{\alpha\in\Gamma} S_\alpha$, each S_α is indiscrete ($\tau = \{\varnothing, S\}$), describe the open subsets of S.
6. Let $X = \prod_{\alpha\in\Gamma} S_\alpha$ and $Y = \prod_{\gamma\in\Phi} T_\gamma$ be two product spaces where Γ and Φ have the same cardinality confirmed by the one-to-one function $q : \Gamma \to \Phi$. For each α, S_α and $T_{q(\alpha)}$ are homeomorphic topological spaces. Show that X and Y are homeomorphic.

7. Suppose we are given a topological space (S, τ_S) and a family of topological spaces, $\{(T_\alpha, \tau_\alpha) : \alpha \in \Gamma\}$. Suppose $\mathscr{F} = \{f_\alpha : \alpha \in \Gamma\}$ is a family of functions, $f_\alpha : S \to T_\alpha$, where each f_α maps its domain S into T_α. Let
$$\mathscr{B} = \{f_\alpha^{\leftarrow}[U] : (\alpha, U) \in \Gamma \times \tau_\alpha\}$$
Show that $\mathscr{B}$ is a base for open sets of S if and only if $\mathscr{F}$ separates points and closed sets in S.

8. Suppose $U \subseteq S$ and $V \subseteq T$. Show that $\text{int}_{S\times T}(U \times V) = \text{int}_S U \times \text{int}_T V$.

9. Show that, if the product space, $\prod_{\alpha\in\Gamma} S_\alpha$, is first countable, then S_α is first countable for each $\alpha \in \Gamma$.

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8 / The quotient topology.

Abstract. *In this section we will present a method to topologize the range, T, of a function, f, whose domain, S, is a topological space. The topology on the range is referred to as the "quotient topology induced by f". When the function f is used for this purpose, it is referred to as the associated quotient map.*

8.1 The strong topology induced by a function.

We have previously discussed a function, f, with an untopologized domain, S, which is mapped to a topological space (T, τ_T). We topologized S in such a way that guarantees the continuity of the function $f : S \to T$ on S. To do this, we must be sure that S is provided with enough open sets so that $f : S \to T$ is continuous. The easiest way to do this is to assign to S the discrete topology, since, on such a space any function will be continuous on S. But, ideally, we preferred a topology which is custom-made for f and τ_T. We then opted for the *weak topology induced by* f and τ_T. Namely,

$$\tau_S = \{U \subseteq S : U = f^{\leftarrow}[V], \text{ for some open } V \text{ in } T\ \}$$

In this section, we will work the other way around. We are given a topological space, S, and a function $f : S \to T$. We wish to assign to T a topology, τ_T, that will guarantee that f is continuous on S.

Again, we could take the easy way out by assigning to T the indiscrete topology, $\{\varnothing, T\}$. It only has one non-empty open set, T, pulled back by f to the open set, $S \in \tau_S$. We again obtain continuity, but one which is independent of the function, f. We opt for choosing a topology on T which is custom made for the given function, f, and the topology on the domain. That is, we will choose the *strongest topology on* T that will guarantee continuity of f on the given topological space (S, τ_S). With this mind, we present the following formal definition.

Definition 8.1 Let $f : S \to T$ be a function mapping the topological space (S, τ_S) *onto* the set T. We assign to the set T the following topology

$$\tau_f = \{U \subseteq T : f^{\leftarrow}[U] \text{ is open in } S\ \}$$

The set, τ_f, is referred to as the *quotient topology induced by* f and τ_S. When T is equipped with the quotient topology, then (T, τ_f) is referred to as the *quotient space induced by* f and $f : S \to T$ is referred to as its associated *quotient map.*[1]

The set τ_f is referred to as being the *strongest (or finest) topology on* T that will guarantee continuity of $f : S \to T$ on the given topological space (S, τ_S).

Since $f^{\leftarrow}$ respects both infinite unions and intersections of sets, then τ_f is a well-defined topology on T. Also, since τ_f contains precisely those sets which are pulled back to some set in τ_S, and no other sets, then τ_f is indeed the largest topology that guarantees continuity for f on S. The quotient topology, τ_f, induced by f is then unique.

It is important for the reader to notice that, in the above definition, we declare the function $f : S \to T$ to be *onto* T. This fact may be relevant in some of the proofs that follow. If $f : S \to T$ was not declared to be *onto* T, we would at least have to modify our definition of quotient topology to

$$\tau_f = \{U \subseteq f[S] : f^{\leftarrow}[U] \in \tau_S\}$$

What we have discussed up to now has to do with topologizing a set T with the help of a function $f : S \to T$ mapping a topological S space onto to T, in such a way that the topology defined on T guarantees that the function is continuous on S. The space T is then said to be a *quotient space induced by the quotient map* f. Suppose, on the other hand that we are given two topological spaces (S, τ_S) and (T, τ_T) and a continuous function, f, mapping S onto T. If we are given no more information about τ_T, we may wonder whether the topology on T is the quotient topology induced by f. It would be useful to have a few tools to determine this. The following theorem shows that there are various ways of recognizing a quotient topology induced by a given function.

Theorem 8.2 Suppose (S, τ_S) and (T, τ_T) are topological spaces and $f : S \to T$ is a continuous function *onto* T.

(a) If $f : S \to T$ is an open map, then τ_T is the quotient topology on T induced by f. That is, $\tau_T = \tau_f$.

(b) If $f : S \to T$ is a closed map, then τ_T is the quotient topology on T induced by f. That is, $\tau_T = \tau_f$.

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[1] Some authors use the terms "identification topology" instead of *quotient topology* and "identification map" instead of *quotient map.*

(c) If there is a continuous function $g : T \to S$ such that $(f \circ g)(x) = x$ on T, then $\tau_T = \tau_f$.

Proof: We are given that $f : S \to T$ is a continuous function mapping S *onto* T with topology τ_T. To show that τ_T is the quotient topology induced by f, it suffices to show that τ_T is the strongest topology on T, τ_f that will guarantee continuity to this function, f. Since $f : S \to T$ is continuous, and τ_f is the largest topology on T for which f is continuous, then $\tau_T \subseteq \tau_f$. For each of the three parts it will then suffice to show that $\tau_f \subseteq \tau_T$.

(a) We are given that $f : S \to T$ is open.
We claim that $\tau_f \subseteq \tau_T$. Let $U \in \tau_f$ in T. By definition of τ_f, $f^{\leftarrow}[U]$ is open in S. Since f is both open and onto, then $f[f^{\leftarrow}[U]] = U \in \tau_T$ in T. So $\tau_f \subseteq \tau_T$. Hence $\tau_T = \tau_f$.

(b) We are given that $f : S \to T$ is a closed continuous function.
We claim that $\tau_f \subseteq \tau_T$. Let $U \in \tau_f$ in T. By definition of τ_f, $f^{\leftarrow}[U]$ is open in S. Since f is both closed and onto, then, by hypothesis, $f[S \backslash f^{\leftarrow}[U]]$ is closed in T. See that

$$f[S \backslash f^{\leftarrow}[U]] = f[f^{\leftarrow}[T \backslash U]] = T \backslash U$$

Since $T \backslash U$ is closed in T, then U is open in τ_T in T. So $\tau_f \subseteq \tau_T$. Hence $\tau_T = \tau_f$.

(c) We are given that $g : T \to S$ is continuous such that $(f \circ g)(x) = x$ (where $f \circ g : T \to T$).
We claim that $\tau_f \subseteq \tau_T$. Suppose $U \in \tau_f$. Then, by definition of τ_f, $f^{\leftarrow}[U]$ is open in S. We have

$$\begin{aligned} U &= \{x \in T : (f \circ g)(x) = x \in U\} \\ &= (f \circ g)^{\leftarrow}[U] \\ &= g^{\leftarrow}[f^{\leftarrow}[U]] \\ \Rightarrow &\ U \text{ is open in } T \text{ since } g \text{ and } f \text{ are continuous.} \\ \Rightarrow &\ U \in \tau_T \end{aligned}$$

Hence $U \in \tau_T$. So $\tau_f \subseteq \tau_T$, as claimed
Then $\tau_f = \tau_T$, as required.

Part (a) of the previous theorem states that, given a function $f : S \to T$ mapping (S, τ_S) onto (T, τ_T), if f is open then τ_T is the quotient topology induced by f. Some curious readers will certainly wonder:

If the set T is topologized by the quotient function, $\varphi : S \to T$, mapping the topological space (S, τ_S) onto T is φ necessarily an open function?

Well, if U is any open subset of S and τ_T is the corresponding quotient topology on T, then $\varphi[U]$ is open in T only if $\varphi^{\leftarrow}[\varphi[U]]$ is open in S.

Example 1. Consider the Cartesian product $\mathbb{R} \times \mathbb{R}$ equipped with the product topology. Recall that $\pi_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is the projection map defined as $\pi_1(a, b) = a$. Since we have previously shown that the projection map $\pi_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is an open function then, by the above theorem, $\mathbb{R}$ equipped with the usual topology is the quotient topology induced by the quotient map, π_1.

8.2 An equivalence relation induced by a function f.

Any function, f, mapping a set S onto a set T can be used to partition the domain into subsets we call "fibres in S induced by the function f". We will first formally define this set theoretic notion.

Definition 8.3 Let $f : S \to T$ be a function mapping a set S *onto* a set T. If $w \in T$, we will refer to $f^{\leftarrow}[\{w\}]$ as the *fibre of w under the map f*. Fibres in the domain are the preimages of singleton sets with respect to f. [2]

Let $f : S \to T$ be a function which maps the topological space (S, τ_S) *onto* the quotient space, (T, τ_f). Using this function we will construct a new set by defining a relation R_f on S as follows:

$$[\, u \text{ is related to } v \,] \quad \Leftrightarrow \quad [\, f(u) = f(v) \text{ in } T \,]$$

The phrase, "u is related to v" in S, can be more succinctly expressed as,

$$uR_fv \qquad \text{or} \qquad (u, v) \in R_f$$

This essentially means that u and v are related if and only if they both belong to the same fibre under the map $f : S \to T$.[3] The relation, R_f,

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[2] The word "fibre" is also written as "fiber". It is usually interpreted in this way in the field of set theory. But it can have an entirely different meaning in other mathematical fields. Its interpretation is determined by the context.

[3] In this context a fibre is a set theoretic concept and is independent of the topologies involved.

on S is easily seen to be reflexive, symmetric and transitive, and so R_f is an *equivalence relation* on S. We will denote an equivalence class of x under R_f by

$$S_x = \{y \in S : xR_fy\}$$

and the set of all such equivalence classes in S by

$$S/R_f = \{\, S_x : x \in S \,\}$$

Each element, S_x, is a fibre under, f, and the set, S/R_f, can be viewed as the set of all fibres in S under f. In set theory, S/R_f, is normally called the

quotient set of S induced by R_f.

Just like the set of all fibres of a function, $f : S \to T$, partitions the domain, we see that the set of all equivalence classes partitions the set S. By this we mean that the elements of S/R_f are pairwise disjoint subsets which cover all of S. One should remember that, when S_x is in S, it is a subset of S, but, whenever S_x is in S/R_f, it is viewed as one of its elements.

There is a "natural" function, $\theta : S \to S/R_f$, defined as

$$\theta(x) = S_x$$

mapping the points of the topological space (S, τ_S) onto the "elements" of the set S/R_f. One might more figuratively say that θ collapses each fibre, $f^{\leftarrow}(x)$, in S down to a unique element, S_x, in S/R_f, or, equivalently, maps fibres in S to singleton sets, $\{S_x\}$, in S/R_f.

As long as no topology is declared on S/R_f our discussion remains within the bounds of set theory. We will topologize S/R_f as being the

quotient space induced by the function, $\theta : S \to S/R_f$

a function we can now refer to as a *quotient map.* So S/R_f will be equipped with the quotient topology, τ_θ. More specifically

$$\tau_\theta = \{U : \theta^{\leftarrow}[U] \text{ is open in } S\}$$

The topology on S/R_f will be the strongest topology that guarantees the continuity of $\theta : S \to S/R_f$.

We now have two quotient spaces induced by two functions with domain (S, τ_S):

$$\begin{aligned} f : S &\to (T, \tau_f) \\ \theta : S &\to (S/R_f, \tau_\theta) \end{aligned}$$

An insightful reader may already have a feeling that the two quotient spaces are topologically the same. To some readers this may appear so obvious that, to them, this requires no proof. To confirm this, we have to show that they are linked by some homeomorphism. We will connect S/R_f to T by defining a third function, $\phi_f : S/R_f \to T$ as

$$\phi_f(S_x) = f(x) \qquad \text{(Where } S_x = \theta(x)\text{.)}$$

which maps points of S/R_f to points of T (remembering that f is *onto* T and so every element of T can be represented by $f(x)$ for some x in S.) The following diagram illustrates the relationship between S, S/R_f and T.

$$\begin{tikzcd}
S \arrow[rr, "f"] \arrow[dr, "\theta"'] & & T \\
& S/R_f \arrow[ur, "\phi_f"'] &
\end{tikzcd}$$

This allows us to express the function, f, as a composition of functions $\phi_f \circ \theta = f$. The following definition refers to expressions of the form $f^{\leftarrow}[f[U]]$. One should keep in mind that, even though it may occur that $U = f^{\leftarrow}[f[U]]$, it is always the case that $U \subseteq f^{\leftarrow}[f[U]]$.

In the theorem below we will show that ...

$$\phi_f : S/R_f \to T \text{ maps } S/R_f \text{ homeomorphically onto } T$$

Theorem 8.4 Suppose $f : S \to T$ is a function on S and U is a non-empty subset of T and $\theta : S \to S/R_f$ is the natural map (collapsing the fibres of f down to a point). Then

$$f^{\leftarrow}[U] = \theta^{\leftarrow}[\theta\ [f^{\leftarrow}[U]]]$$

Proof: Since $f^{\leftarrow}[U] \subseteq \theta^{\leftarrow}[\theta\ [f^{\leftarrow}[U]]]$ we need only show inclusion in the opposite direction.

$$\begin{aligned}
x \in \theta^{\leftarrow}[\theta\ [f^{\leftarrow}[U]]] &\Rightarrow \theta(x) \in \theta\,[f^{\leftarrow}[U]] \\
&\Rightarrow \theta(x) \in \{\theta(y) : y \in f^{\leftarrow}[U]\} \\
&\Rightarrow S_x \in \{S_y : f(y) \in U\} \\
&\Rightarrow f(x) \in U \\
&\Rightarrow x \in f^{\leftarrow}[U]
\end{aligned}$$

Then $\theta^{\leftarrow}[\,\theta\,[\,f^{\leftarrow}[U]\,]\,] \subseteq f^{\leftarrow}[U]$.
Then

$$\theta^{\leftarrow}[\,\theta\,[\,f^{\leftarrow}[U]\,]\,] = f^{\leftarrow}[U]$$

Theorem 8.5 Suppose (S, τ_S) is a topological space. Let (T, τ_f) and $(S/R_f, \tau_\theta)$ be two quotient spaces induced by the quotient maps $f : S \to T$ and $\theta : S \to S/R_f$, where $\theta(x) = S_x$. Then the function, $\phi_f : S/R_f \to T$ defined as

$$\phi_f(S_x) = f(x)$$

maps S/R_f homeomorphically onto T.

Proof: We are given that $f : S \to T$ and $\theta : S \to S/R_f$ and are quotient maps, hence are continuous on S. We are required to show that the function, $\phi_f : S/R_f \to T$, defined as, $\phi_f(S_x) = (\phi_f \circ \theta)(x) = f(x)$, is a homeomorphism.

We claim that ϕ_f is one-to-one and onto T: See that, since f is onto T, ϕ_f is onto T. Also, if $S_x \neq S_y$, then $f(x) \neq f(y)$, which implies $\phi_f(x) \neq \phi_f(y)$; so ϕ_f is one-to-one and onto.

We claim that ϕ_f is continuous on S/R_f: Let U be open in T. Since f is continuous $f^{\leftarrow}[U]$ is open in S.

$$\begin{aligned}
\phi_f^{\leftarrow}[U] &= \{S_x : \phi_f(S_x) \in U\} \\
&= \{S_x : \phi_f(\theta(x)) \in U\} \\
&= \{\theta(x) : f(x) \in U\} \\
&= \{\theta(x) : x \in f^{\leftarrow}[U]\} \\
&= \theta[f^{\leftarrow}[U]]
\end{aligned}$$

We have shown above that $\theta^{\leftarrow}[\,\theta\,[\,f^{\leftarrow}[U]\,]\,] = f^{\leftarrow}[U]$. Since $\theta : S \to S/R_f$ is a quotient map and $f^{\leftarrow}[U]$ is open, then $\theta\,[\,f^{\leftarrow}[U]\,]$ is open. So $\phi_f^{\leftarrow}[U]$ is open. This establishes the claim that ϕ_f is continuous.

We claim that ϕ_f is open on S/R_f: From the diagram above we know that $f = \phi_f \circ \theta$. So $\theta = \phi_f^{\leftarrow} \circ f$. Since θ is continuous, then $(\phi_f^{\leftarrow} \circ f) : S \to S/R_f$ is continuous on S. Let U is an open subset of S/R_f.

$$\begin{aligned}
\theta^{\leftarrow}[U] &= (\phi_f^{\leftarrow} \circ f)^{\leftarrow}[U] \in \tau_S \\
&\Rightarrow f^{\leftarrow}[(\phi_f^{\leftarrow})^{\leftarrow}[U]] \in \tau_S \\
&\Rightarrow f^{\leftarrow}[\phi_f[U]] \in \tau_S
\end{aligned}$$

Since T has the quotient topology induced by f, $\phi_f[U]$ is open in T. We have shown that ϕ_f is an open map, as claimed.

We have shown that the one-to-one onto function, $\phi_f : S/R_f \rightarrow T$, is both continuous and open, hence it maps S/R_f homeomorphically onto T.

We summarize the main ideas behind the above result. If we are given a topological space, (S, τ_S), and a function f mapping S onto a set T we have the two main ingredients necessary to topologize T with the quotient topology, τ_f. We do so in a way that guarantees continuity of the function, $f : S \rightarrow T$. The fibres of the function, f, covers its domain. By collapsing, via the map θ, each fibre down to a point we create another set, S/R_f, and topologize it with the quotient topology, τ_θ. The expression "identifying the points of the fibres" is also of common usage. This new topological space, $(S/R_f, \tau_\theta)$ has been proven to be a homeomorphic copy of (T, τ_f).

Saturated sets relative to a function.

We take this opportunity to introduce the notion of "saturated sets" relative to a function.

Definition 8.6 Let $g : S \rightarrow T$ be a function mapping the set S onto the set T. A subset, $U \subset S$, is said to be

g-saturated

or just "saturated relative to the function g", whenever U is the *complete* inverse image of some subset V in T. Equivalently, U is g-saturated if and only if $U = g^{\leftarrow}(g(U))$. Equivalently, U is g-saturated if U is the union of fibres of g.

Suppose, more specifically, that $g : S \rightarrow T$ is a function mapping the topological space (S, τ_S) *onto* a topological space (T, τ_T). A subset, $U \subset S$, is said to be

open g-saturated

whenever U is open in S and is the *complete* inverse image of some open subset V in T. Equivalently, U is *open g-saturated* if and only if U is open in S and $U = g^{\leftarrow}(g[U])$ where $g[U]$ is open in T.

Example 2. Consider the projection map $\pi_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined as $\pi_1(a, b) = a$ viewed as a quotient map. The subset $U = (0, 1) \times \mathbb{R}$ of $\mathbb{R} \times \mathbb{R}$ is open π_1-saturated since $(0, 1) \times \mathbb{R}$ is open in $\mathbb{R} \times \mathbb{R}$ and

$$(0,1) \times \mathbb{R} \;=\; \pi_1^{\leftarrow}[(0,1)] \;=\; \pi_1^{\leftarrow}[\pi_1[(0,1) \times \mathbb{R}]]$$

8.3 Decomposition spaces.

We have seen that the quotient space induced by $\theta : S \to S/R_f$ involves the decomposition of the set S into non-overlapping sets, $\{S_x\}$. We will now slightly generalize this notion of "decomposition of a set". Suppose we are given a topological space (S, τ_S). Rather than partition S into equivalence classes, each of which contains all elements which belong to a particular fibre, S_x, of a function f, we can also partition the set S in an arbitrary way. By this we mean that we express S as the union of non-intersecting subsets. We will denote this set of subsets by, $\mathscr{B}_S$. Each element x of S belongs to some element, labeled D_x, of $\mathscr{D}_S$. Note that, if $y \in D_x$, then D_x and D_y are simply different labels for the same element of $\mathscr{D}_S$. The set

$$\mathscr{D}_S = \{D_x : x \in S\}$$

is reminiscent of a quotient set whose elements are equivalent classes of some relation R on S. We can similarly define a function

$$\theta : S \to \mathscr{D}_S$$

on the topological space, S, where

$$\theta(u) = D_u$$

the unique element of $\mathscr{D}_S$ which contains u. Then $\theta^{\leftarrow}(D_u) = \{x \in S : x \in D_u\} = D_u \subseteq S$. The set, $\mathscr{D}_S$, is not yet topologized, but we know of a procedure to topologize it, since it is, after all, the range of a function θ on S. We can equip $\mathscr{D}_S$ with the quotient topology,

$$\tau_\theta = \{U \subseteq \mathscr{D}_S : \theta^{\leftarrow}[U] \text{ is open in } S\}$$

induced by θ. We formally define the concepts we have just presented.

Definition 8.7 Let (S, τ_S) be a topological space. If $\mathscr{D}_S$ is collection of pairwise disjoint subsets of S such that every element of S belongs to some element of $\mathscr{D}_S$, then we say $\mathscr{D}_S$ is a

“*decomposition of* S”

The function, $\theta : S \to \mathscr{D}_S$, defined as, $\theta(u) = D_x$ if and only if $u \in D_x$, mapping every element of S onto $\mathscr{D}_S$ is called the

natural map

or *decomposition map of* S *onto* $\mathscr{D}_S$. The function, θ, is also referred to as the *identification map.* The function, θ, is said to *identify the elements of the subset* $D_x \subseteq S$. If τ_θ is the quotient topology induced on $\mathscr{D}_S$ by θ, then we say that $(\mathscr{D}_S, \tau_\theta)$ is a

“*decomposition space* or *quotient space*”

Why does all of this sound familiar? Recall that, if $f : S \to T$, then the function, f, decomposes its domain, S, into fibres. In this case, $(\mathscr{D}_S, \tau_\theta)$ is a copy of $(S/R_f, \tau_\theta)$.

The following theorem will help recognize the open subsets of the decomposition space, $\mathscr{D}_S$. It states that open subsets $\mathscr{U} = \{D_x : D_x \in \mathscr{U}\}$ of $\mathscr{D}_S$ correspond to open θ-saturated subsets $\cup\{D_x : D_x \in \mathscr{U}\}$ of S.

Theorem 8.8 Suppose $(\mathscr{D}_S, \tau_\theta)$ is a decomposition space of the topological space (S, τ_S) and let $\mathscr{U} = \{D_x : x \in I \subseteq S\}$ be a subset of $\mathscr{D}_S$. Then $\mathscr{U}$ is open in $\mathscr{D}_S$ if and only if

$$\cup\{D_x : x \in I\}$$

is open in S.

Proof: We are given that $\mathscr{U} = \{D_x : x \in I \subseteq S\}$ be a subset of $\mathscr{D}_S$.

($\Rightarrow$) Suppose $\mathscr{U}$ is open in $\mathscr{D}_S$.

Then $\theta^{\leftarrow}[\mathscr{U}] = \{x \in S : \theta(x) = D_x \in \mathscr{U}\} = \{x \in S : x \in I\}$ is open in S. We are required to show that $\cup\{D_x : x \in I\}$ is open in S.

$$\begin{aligned} \theta^{\leftarrow}[\mathscr{U}] &= \theta^{\leftarrow}[\cup\{\{D_x\} : x \in I\}] \\ &= \cup\{\theta^{\leftarrow}[\{D_x\}] : x \in I\} \\ &= \cup\{D_x \subseteq S : x \in I\} \\ & \text{Is open since } \theta \text{ is continuous.} \end{aligned}$$

($\Leftarrow$) Suppose $\cup\{D_x \subseteq S : x \in I\}$ is open in S. We are required to show that $\mathscr{U} = \{D_x : x \in I \subseteq S\}$ is open in $\mathscr{D}_S$. To show that $\mathscr{U}$ is open in $\mathscr{D}_S$, it suffices to show that $\theta^{\leftarrow}[\mathscr{U}]$ is open in S (since $\mathscr{D}_S$ is equipped with the quotient topology induced by θ).

$$\begin{aligned} \cup\{D_x \subseteq S : x \in I\} &= \cup\{\theta^{\leftarrow}[\{D_x\}] : x \in I\} \\ &= \theta^{\leftarrow}[\cup\{ D_x : x \in I\}] \\ &= \theta^{\leftarrow}[\mathscr{U}] \end{aligned}$$

Since $\cup\{D_x : x \in I\} = \theta^{\leftarrow}[\mathscr{U}]$ is open, then $\mathscr{U}$ is open.

Based on the definition and the above theorem, it is clear that

> *... every decomposition space is a quotient space induced the quotient map θ.*

Example 3. For each real number $r \geq 0$, we define the subset

$$C_r = \{(x, y) : \sqrt{x^2 + y^2} = r\}$$

of $\mathbb{R} \times \mathbb{R}$. Then the collection

$$\mathscr{D} = \{C_r : r \geq 0\}$$

partitions the set $\mathbb{R} \times \mathbb{R}$ and so forms a decomposition of $\mathbb{R} \times \mathbb{R}$. We define the function $\theta : \mathbb{R} \times \mathbb{R} \to \mathscr{D}$ as

$$\theta(x, y) = C_r \text{ provided } \sqrt{x^2 + y^2} = r$$

so that each point of $\mathbb{R} \times \mathbb{R}$ is mapped to the only element in $\mathscr{D}$ which contains it. So θ is a natural map. We topologize the decomposition $\mathscr{D}$ of $\mathbb{R} \times \mathbb{R}$ by defining its open subsets as

$$\tau_\theta = \{\mathscr{U} \subseteq \mathscr{D} : \cup\{C_r : C_r \in \mathscr{U}\} \in \tau_{\mathbb{R}\times\mathbb{R}}\}$$

For example, if $k > 0$, $\mathscr{U} = \{C_r : k - \varepsilon < r < k + \varepsilon\}$ is an open neighborhood of C_k in the decomposition space $\mathscr{D}$ with natural map θ.

Upper semicontinuous decomposition spaces.

We introduce immediately what can sometimes be a very useful type of decomposition space which will be called upon in a chapter further on in the text.

Definition 8.9 Suppose S is a topological space and $\mathscr{D}$ is a decomposition of S equipped with the quotient topology induced by the natural map $\theta : S \to \mathscr{D}$. Suppose V is an open subset of S which is θ-saturated. By this we mean that $V = \theta^{\leftarrow}[W]$ where $W = \theta[V]$ is an open subset of $\mathscr{D}$. We say that $\mathscr{D}$ is an

upper semicontinuous decomposition space

if and only if for any element A in $\mathscr{D}$ which is contained in a subset U of S there exists a θ-saturated open set V in S such that $A \subseteq V \subseteq U$.

Example 4. In the previous example, we introduced a decomposition space of $\mathbb{R} \times \mathbb{R}$

$$\mathscr{D} = \{C_r : r \geq 0\}$$

where

$$C_r = \{(x, y) : \sqrt{x^2 + y^2} = r\}$$

with natural map $\theta(x, y) = C_r$ provided $\sqrt{x^2 + y^2} = r$.

We can confirm that this is an *upper semicontinuous decomposition space* as follows. Suppose $C_k \in \mathscr{D}$ where $C_k \subseteq U$ where U is an open subset of $\mathbb{R} \times \mathbb{R}$. Then, for every point (x, y), there is an open ball $B_\varepsilon(x, y)$ such that $B_\varepsilon(x, y) \subseteq U$. Then $\{B_\varepsilon(x, y) : (x, y) \in C_k\}$ forms an open cover of the compact set C_k. Then there is an open subcover $\{B_{\varepsilon_i}(x_i, y_i) : i = 1 \text{ to } n\}$ of C_k. Let $\delta = \min\{\varepsilon_i : i = 1 \text{ to } n\}$. Let $V = \{C_r : k - \delta < r < k + \delta\}$ an open θ-saturated subset of $\mathbb{R} \times \mathbb{R}$. Then $C_k \subseteq V \subseteq U$. So the decomposition $\mathscr{D} = \{C_r : r \geq 0\}$ is upper semicontinuous.

Theorem 8.10 Suppose S is a topological space and $\mathscr{D}$ is a decomposition of S equipped with the quotient topology induced by the natural map $p : S \to \mathscr{D}$. Then p is a closed natural map if and only if $\mathscr{D}$ is an upper semicontinuous decomposition space.

Proof: ($\Rightarrow$) Suppose $p : S \to \mathscr{D}$ is a closed map. Suppose $A \in \mathscr{D}$ and U is an open subset of S such that $A \subseteq U$. By hypothesis, since $S \backslash U$ is open in S, $p[S \backslash U]$ is closed in $\mathscr{D}$. Since p is continuous, $p^{\leftarrow}[p[S \backslash U]]$ is a closed p-saturated subset of S. Then $V = S \backslash p^{\leftarrow}[p[S \backslash U]]$ is a p-saturated open subset of S. We can easily verify that $A \subseteq V \subseteq U$. So $\mathscr{D}$ is an upper semicontinuous decomposition of S.

($\Leftarrow$) Suppose $\mathscr{D}$ is an upper semicontinuous decomposition of S with natural map $p : S \to \mathscr{D}$. Suppose Z is a closed subset of S. We are required to show that $p[Z]$ is a closed subset of $\mathscr{D}$. To do this it suffices to show that $\mathscr{D} \backslash p[Z]$ is open in $\mathscr{D}$. Let $A \in \mathscr{D} \backslash p[Z]$. Then $A \subseteq S \backslash Z$. By hypothesis, there exists an open p-saturated subset V of S such that $A \subseteq V \subseteq S \backslash Z$. Since V is an open p-saturated subset of S there is an open subset W of $\mathscr{D}$ such that $p^{\leftarrow}[W] = V$. Then $A \in p[V] = W \subseteq \mathscr{D} \backslash p[Z]$. That is, A is an element of the open subset W which misses $p[Z]$. We can then conclude that $p[Z]$ is closed, hence p is a closed map. As required.

Theorem 8.11 Let $\pi : S \to T$ be quotient map mapping S onto T. Then π is a closed map if and only if the collection $\mathscr{D} = \{\pi^{\leftarrow}(x) : x \in T\}$ is an upper semicontinuous decomposition of the space S .

Proof: ($\Rightarrow$) Suppose $\pi : S \to T$ is a closed map mapping a topological space S onto the space T. We are required to show that the partition $\mathscr{D} = \{\pi^{\leftarrow}(x) : x \in T\}$ of S forms an upper semicontinuous decomposition of the space S. Suppose $t \in \pi[S]$. Then $\pi^{\leftarrow}(t) \subseteq S$. Suppose U is an open subset of S such that $\pi^{\leftarrow}(t) \subseteq U$. We are required to produce an open π-saturated subset V of S such that $\pi^{\leftarrow}(t) \subseteq V \subseteq U$. By hypothesis, $\pi[S \backslash U]$ is closed in $\pi[S]$. Since π is continuous, $\pi^{\leftarrow}[\pi[S \backslash U]]$ is a closed π-saturated subset of S. Then $V = S \backslash \pi^{\leftarrow}[\pi[S \backslash U]]$ is an open π-saturated subset of S. Then $\pi^{\leftarrow}(t) \subseteq V \subseteq U$. So the decomposition $\mathscr{D} = \{\pi^{\leftarrow}(x) : x \in T\}$ of S forms an upper semicontinuous decomposition of S.

($\Leftarrow$) Suppose $\pi : S \to T$ is a quotient map. Then the set $\{\pi^{\leftarrow}(x) : x \in T\}$ is a decomposition of the space S.

Suppose Z is a closed subset of S. We are required to show that $\pi[Z]$ is a closed subset of $\pi[S]$. To do this it suffices to show that $\pi[S]\backslash\pi[Z]$ is open in $\pi[S]$. Let $t \in \pi[S]\backslash\pi[Z]$. Then $\pi^{\leftarrow}(t) \subseteq S\backslash Z$. By hypothesis, there exists an open π-saturated subset V of S such that $\pi^{\leftarrow}(t) \subseteq V \subseteq S\backslash Z$. Since V is an open π-saturated subset of S there is an open subset W of $\pi[S]$ such that $\pi^{\leftarrow}[W] = V$. Then $t \in \pi[V] = W \subseteq \pi[S]\backslash\pi[Z]$. That is, t is an element of the open subset W which misses $\pi[Z]$. We can then conclude that $\pi[Z]$ is closed, hence π is a closed quotient map. As required.

Example 5. Let $\pi : S \to T$ be a closed continuous function mapping the topological space S onto the space T. Then, by the theorem above, $\mathscr{D} = \{\pi^{\leftarrow}(x) : x \in T\}$ forms an upper semicontinuous decomposition of S. Let $f \in C(\mathscr{D})$. We define a function $f^* : T \to \mathbb{R}$ as $f^*(x) = f[\pi^{\leftarrow}(x)]$ (where $\pi^{\leftarrow}(x) \in \mathscr{D}$). Show that f^* is continuous on T.

Solution: Since $\pi : S \to T$ is a closed map, then T is equipped with the quotient topology induced by the closed quotient map π. We are given that $f \in C(\mathscr{D})$ and, for $x \in T$, that $f^*(x) = f[\pi^{\leftarrow}(x)]$.

Let U be an open subset of $\mathbb{R}$. It suffices to show that $f^{*\leftarrow}[U]$ is open in T. Let $z \in f^{*\leftarrow}[U] \subseteq T$. So it suffices to produce an open subset W of T such that $z \in W \subseteq f^{*\leftarrow}[U]$.

$$\begin{array}{rcl} \pi^{\leftarrow}(z) & \subseteq & \pi^{\leftarrow}f^{*\leftarrow}[U] \\ & = & \pi^{\leftarrow}[(f \circ \pi^{\leftarrow})^{\leftarrow}[U]] \\ & = & \pi^{\leftarrow}[(\pi \circ f^{\leftarrow})[U]] \\ & = & f^{\leftarrow}[U] \end{array}$$

Then $\pi^{\leftarrow}(z)$ is a subset of the open subset $f^{\leftarrow}[U]$ of S. Since π is a closed map then, by the above theorem, there exists an open π-saturated subset V such that

$$\pi^{\leftarrow}(z) \subseteq V \subseteq f^{\leftarrow}[U] = \pi^{\leftarrow}f^{*\leftarrow}[U]$$

Then there is an open subset W of T such that $\pi^{\leftarrow}[W] = V$. Then

$$\begin{aligned} z &= \pi[\pi^{\leftarrow}(z)] \\ &\subseteq \pi[V] \\ &= \pi[\pi^{\leftarrow}[W]] \\ &= W \\ &\subseteq \pi[f^{\leftarrow}[U]] \\ &= \pi[\pi^{\leftarrow} f^{*\leftarrow}[U]] \\ &= f^{*\leftarrow}[U] \end{aligned}$$

Then $f^{*\leftarrow}[U]$ is open in T and so f^* is continuous on T, as required.

Example 6. Suppose we are given the subset, $S = [0, 2\pi] \times [0, 2\pi]$ of $\mathbb{R}^2$. We will decompose S as follows: For each point $(x, 0)$ on the line $[0, 2\pi] \times \{0\}$ let

$$D_{(x,0)} = \{(x, 0), (2\pi - x, 2\pi)\}$$

So for each $x \in [0, 2\pi]$, $D_{(x,0)}$ and $D_{(2\pi-x,2\pi)}$ represent the same element of the decomposition, $\mathscr{D}_S$. For example, $D_{(0,0)} = D_{(2\pi,2\pi)} = \{(0, 0), (2\pi, 2\pi)\}$.

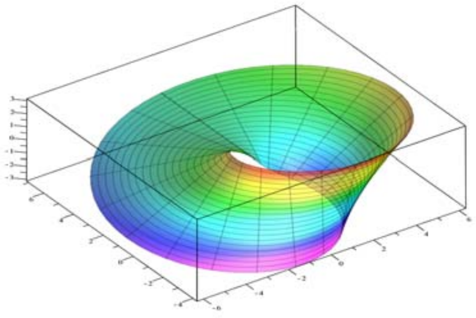

Figure 3: Topological representation of the Mobius strip

For each $(x, y) \notin [0, 2\pi] \times \{0\} \cup [0, 2\pi] \times \{2\pi\}$ let

$$D_{(x,y)} = \{(x, y)\}$$

Each subset of the form $D_{(x,0)}$ is a subset of S which contains two points. All other subsets of the decomposition are singleton sets. The decomposition space, $\mathscr{D}_S$, of the subspace S obtained is referred to as the *Möbius strip.*[4] See the figure.

The two lines $L_1 = \{(x, 0) : 0 \leq x \leq 2\pi\}$ and $L_2 = \{(x, 2\pi) : 0 \leq x \leq 2\pi\}$ are collapsed together (after inverting one of the lines) to form the mobius strip.

Without the inversion of one of the lines, the decomposition space becomes a topological representation of a cylindrical shell.

Moore plane decomposition

Example 7. Let (S, τ_S) denote the Moore plane. (Review the description of the Moore plane by looking over the example on page 87.) Let

$$F = \mathbb{R} \times \{0\}$$

and $W = S \setminus F$. We can then express S as the disjoint union of the sets F and W. Let

$$\mathscr{D} = \{\{x\} : x \in W\} \cup \{F\}$$

be the decomposition space of S which results from collapsing the subset, F, down to a point and identifying all other points to themselves.

Is $\mathscr{D}$ an upper semicontinuous decomposition of S? Is the natural map $\theta S \to \mathscr{D}$ an open map?

Solution: We are given the Moore plane, (S, τ_S), and the decomposition $\mathscr{D} = \{\{x\} : x \in W\} \cup \{F\}$. Note that S is T_1.

If $\theta : S \to \mathscr{D}$ is the corresponding natural map, then

$$\begin{aligned} \theta(a, b) &= (a, b) \text{ on } W \\ \theta[\mathbb{R} \times \{0\}] &= \{F\} \end{aligned}$$

The quotient topology on $\mathscr{D}$ is, by definition,

$$\tau_\theta = \{U \subseteq \mathscr{D} : \theta^{\leftarrow}[U] \text{ is open in } S\}$$

The open neighborhood base of points, $\{x\} \in \mathscr{D}$ for $x \in W$ is the same as for τ_W.

An open neighborhood for the subset $F = \mathbb{R} \times \{0\} \subseteq S$ is of the form:

$$\cup \{B_{\varepsilon_x}(x, \varepsilon_x) \cup \{(x, 0)\} : x \in \mathbb{R}\}$$

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[4] In reference to August Ferdinand Möbius, 1790-1868. A German mathematician and astronomer

We now describe an open neighborhood base for the element, $F \in \mathscr{D}$.

If $\mathscr{U}$ is an open neighborhood of $F \in \mathscr{D}$, the set $\theta^{\leftarrow}[\mathscr{U}]$ must be an open neighborhood of $F = \mathbb{R} \times \{0\}$ in S. Then $\theta^{\leftarrow}[\mathscr{U}]$ must contain a subset of the form $\cup\{B_{\varepsilon_x}(x, \varepsilon_x) \cup \{(x, 0)\} : x \in \mathbb{R}\}$ which in turn contains $\mathbb{R} \times \{0\}$.

Since the decomposition is clearly upper semicontinuous on W we need only verify this property holds on F.

Let M be an open neighborhood of F in S. Then for each $x \in \mathbb{R}$ there is an ε_x such that $B_{\varepsilon_x}(x, \varepsilon_x) \cup (x, 0) \subseteq M$. Then $U = \cup\{B_{\varepsilon_x}(x, \varepsilon_x) \cup (x, 0) : x \in \mathbb{R}\}$ is a neighborhood of F. Since $\theta[U] = U$ then U is an open θ-saturated neighborhood of F which is contained in M.

We conclude that $\mathscr{D}$ is an upper semicontinuous decomposition.
Is the natural map $\theta : S \to \mathscr{D}$ an open map? It is not open. Consider, for example, the open neighborhood $B_{\varepsilon_4}(4, \varepsilon_4) \cup \{(4, 0)\}$ of the point $(4, 0)$. Then $\theta[B_{\varepsilon_4}(4, \varepsilon_4) \cup \{(4, 0)\}] = B_{\varepsilon_4}(4, \varepsilon_4) \cup \{F\}$. This is not an open neighborhood of F in $\mathscr{D}$

Concepts review.

1. Define the quotient topology induced by a function.

2. Let (S, τ_S) be a topological space. If τ_T is a topology on T for which $f : S \to T$ is continuous, how does τ_T compare with the quotient topology, τ_f, induced by f?

3. Let (S, τ_S) be a topological space. If τ_T is a topology on T for which $f : S \to T$ is a continuous open map, how does τ_T compare with the quotient topology, τ_f, induced by f?

4. If τ_T is a topology on T for which $f : S \to T$ is a continuous closed map, how does τ_T compare with the quotient topology, τ_f, induced by f?

5. What is a *fibre* of a point under a map f?

6. Define the set S/R_f of all equivalence classes induced by a function $f : S \to T$ by referring to a function $\theta : S \to S/R_f$.

7. What topology is defined on S/R_f?

8. How is the function $\phi_f : S/R_f \to T$ defined?
9. If $g : S \to T$ is a function and $U \subseteq S$, what does it mean to say that U is g-saturated?
10. If $g : S \to T$ is a function and $U \subseteq S$ is g-saturated, is there a way to describe the subset U in terms of g?
11. Given $f : S \to T$ and $\theta : S \to S/R_f$ describe the homeomorphism which links S/R_f to T.
12. Describe how to construct a decomposition space, $\mathscr{D}_S$, from a topological space (S, τ_S).

EXERCISES

1. Let (S, τ_S) be a topological space. We define a relation, R, on S as follows: $(u, v) \in R$ if and only if $\text{cl}_S\{u\} = \text{cl}_S\{v\}$.
   (a) Show that R is an equivalence relation on S.
   (b) Let $(S/R, \tau_\theta)$ denote the quotient space induced by the natural map $\theta : S \to S/R$.
      (i) If S_x is an element in S/R, is $\{S_x\}$ necessarily a closed subset of S/R?
      (ii) If S_x and S_y are distinct elements of S/R, does there exist an open set in τ_θ which contains S_x but not S_y?
2. Describe the quotient space, $\mathbb{R}/R$, if $(u, v) \in R$ if and only if $x - y$ is an integer.
3. Suppose $\mathscr{D}_{\mathbb{R}^2}$ is a decomposition of $\mathbb{R}^2$ (with the usual topology) whose elements are circles with center at the origin. Show that the corresponding decomposition space, $(\mathscr{D}_{\mathbb{R}^2}, \tau_\theta)$, and the set of all non-negative real numbers, $\{x \in \mathbb{R} : x \geq 0\}$, equipped with the usual topology are homeomorphic topological spaces.