Appendix E.

The following results are in reference to the proposed example of a regular non-completely regular space other than the standard the Tychonoff corkscrew example discussed on page 245.

Lemma E1. Let S be a topological space which contains disjoint closed subsets H and K which cannot be separated by disjoint open subsets of S. Let V be an open subset of S which contains H. Then $\mathrm{cl}_S V \cap K$ and H cannot be separated by disjoint open subsets sets of S.

Proof: We are given two closed subsets H and K of S which cannot be completely separated by disjoint open subsets of S. Suppose V is any open set in S such that

$$H \subseteq V$$

Suppose there exists disjoint open subset D and E of S such that

$$\begin{aligned} (\mathrm{cl}_S V) \cap K &\subseteq D \\ H &\subseteq E \end{aligned}$$

Since $E \cap D = \varnothing$, then $\mathrm{cl}_S E \cap D = \varnothing$.

Also, since $H \subseteq E \cap V$, $\mathrm{cl}_S(E \cap V) \cap K \neq \varnothing$ (for, if it was empty, $S \backslash \mathrm{cl}_S(E \cap V)$ and $E \cap V$ would be disjoint open subsets of S which separate H and K contradicting the hypothesis).

$$\begin{aligned} \varnothing &\neq \mathrm{cl}_S[E \cap V] \cap K \\ &\subseteq [\mathrm{cl}_S E \cap \mathrm{cl}_S V] \cap K \\ &= \mathrm{cl}_S E \cap [\mathrm{cl}_S V \cap K] \\ &\subseteq \mathrm{cl}_S E \cap D \end{aligned}$$

Then $\mathrm{cl}_S E \cap D \neq \varnothing$, a contradiction. So there cannot exist disjoint open subset E and D of S containing H and $\mathrm{cl}_S V \cap K$.

Lemma E2. Let S be a non-normal topological which contains disjoint closed subsets H and K which cannot be separated by disjoint open subsets of S.

Consider the set

$$Z = (S \times \mathbb{N}) \cup \{\omega\}$$

The set Z is topologized as follows: $S \times \mathbb{N}$ is equipped with the usual product topology, and $S \times \mathbb{N}$ is open in Z. The point, ω, has an open neighborhood base $\mathscr{B} = \{B_n : n = 1, 2, 3, \ldots, \}$ where

$$B_n = Z \backslash (S \times \{1, 2, 3, \ldots n\})$$

$$B_n = \{\omega\} \quad \cup \quad (S \times \{n+1, n+2, n+3, \ldots\})$$

The base $\mathscr{B}$ generates the topological space, (Z, τ_Z). Then

(a) If S is Hausdorff then so is Z.

(b) If S is regular then so is Z.

(c) If S is completely regular then so is Z.

Proof (a): Suppose S is Hausdorff.

Let $(a, n), (b, m)$ be distinct points in $S \times \mathbb{N}$. If $n \neq m$, then $S \times \{n\}$ and $S \times \{m\}$ are disjoint open neighborhoods of (a, n) and (b, m). If $n = m$ then $a \neq b$. Since S is Hausdorff there exists disjoint open neighborhoods U and V in S which separate a and b. Then $U \cap (S \times \{n\})$ and $V \cap (S \times \{m\})$ are disjoint open subsets of Z which contain (a, n) and (b, m). So Z is Hausdorff.

Proof of (b): Suppose S is regular.

Case 1. Let F be a closed subset of Z which does not contain the point ω.

Then there is $n \in \mathbb{N}$ such that $B_n \cap F = \varnothing$. We claim that B_n is clopen in Z. For each $r \leq n$, $S \times \{r\}$ is open in Z. Then $\cup\{S \times \{r\} : r \leq n\}$ is open in Z. Then $Z \backslash \cup\{S \times \{r\} : r \leq n\}$ is closed. Since $Z \backslash \cup\{S \times \{r\} : r \leq n\} = B_n$, then B_n is clopen, as claimed.

Then B_n and $Z \backslash B_n$ are disjoint open neighborhoods of ω and F in Z.

Case 2. Suppose, on the other hand, that F is a closed subset of Z which does not contain (a, d).

Subcase 2.1. Suppose $\omega \notin F$. Then F is a closed subset of $S \times \mathbb{N}$ which does not intersect $\{\omega\}$. There exist disjoint open neighborhoods U and V of F and (a, d) in $S \times \mathbb{N}$. Then U and V are disjoint open neighborhoods in Z which separate F and (a, d) in Z.

Subcase 2.2. Suppose ω belongs to the closed subset F and $(a, d) \notin F$. There exists disjoint open neighborhoods U and V of $F \cap (S \times \mathbb{N})$ and (a, d), respectively, in $S \times \mathbb{N}$. Choose $n \in \mathbb{N}$ such that $B_n \cap V = \varnothing$. Then $B_n \cup U$ and V are disjoint open neighborhoods of F and (a, d) in Z.

The Z is a regular space.

Proof of (c): Suppose S is completely regular.

Case 1. Let F be a closed subset of Z which does not contain the point ω.

Then there exists $m \in \mathbb{N}$ such that $B_m \cap F = \varnothing$. Then B_m and $Z \backslash B_m$ are disjoint clopen subsets of Z which contain $\{\omega\}$ and F respectively.

Then the continuous function $f : Z \to [0,1]$ defined as $f[B_m] = \{0\}$ and $f[Z \backslash B_m] = \{1\}$ separates $\{\omega\}$ and F, as required.

Case 2. Suppose that F is a closed subset of Z which does not contain (a, d).

Subcase 2.1. Suppose $\omega \notin F$.

Since F is closed, there exists a neighborhood B_n such that $B_n \cap (F \cup \{(a, d)\}) = \varnothing$. Both S and $\mathbb{N}$ are completely regular, then so is $S \times \mathbb{N}$. Then there exists a function $f : S \times \mathbb{N} \to [0, 1]$ such that $F \subseteq Z(f)$ and $(a, d) \in Z(f - 1)$. We can define $f[B_n] = \{0\}$. Since B_n is clopen, then $f : Z \to [0, 1]$ is a continuous function on Z which separates F and (a, d).

Subcase 2.2. Suppose ω belongs to the closed subset F and $(a, d) \notin F$. Then $F \cap (S \times \mathbb{N})$ is closed in $S \times \mathbb{N}$. Then there exists a function $f : S \times \mathbb{N} \to [0, 1]$ such that $F \cap (S \times \mathbb{N}) \subseteq Z(f)$ and $(a, d) \in Z(f - 1)$. Let B_n be a clopen neighborhood of ω in Z which misses (a, d). Define $f[B_n] = \{0\}$. Then $f : Z \to [0, 1]$ is a continuous function on Z which separates F and (a, d).

Lemma E3. Let S be a non-normal topological which contains disjoint closed subsets H and K which cannot be separated by disjoint open subsets of S.

Consider the set

$$Z = (S \times \mathbb{N}) \cup \{\omega\}$$

equipped with the topology τ_Z described above in Lemma E2.

We partition the set Z as follows:

$$\begin{aligned} A_1 &= \{\, \{(x, n), (x, n+1)\} : x \in H \text{ and } n \text{ is even} \,\} \\ A_2 &= \{\, \{(y, n), (y, n+1)\} : y \in K \text{ and } n \text{ is odd} \,\} \\ A_3 &= \{\, \{(a, b)\} : a \notin H \cup K \} \;\cup \{\omega\} \end{aligned}$$

So that $\mathscr{D} = A_1 \cup A_2 \cup A_3$ is a decomposition of the space Z.

Let $q : Z \to \mathscr{D}$ be the quotient map associating each point in Z to the unique element in $\mathscr{D}$ which contains it. For example,

$$\begin{aligned} \text{If } x \notin (H \cup K) \;&\Rightarrow\; q((x, 4)) = [(x, 4)] = \{(x, 4)\} \\ \text{If } x \in H \;&\Rightarrow\; q((x, 4)) = [(x, 4)]_H = \{(x, 4), (x, 5)\} \\ \text{If } x \in K \;&\Rightarrow\; q((x, 4)) = [(x, 4)]_K = \{(x, 3), (x, 4)\} \end{aligned}$$

$$\begin{aligned} \text{If } x \notin (H \cup K) \;&\Rightarrow\; q((x, 5)) = [(x, 5)] = \{(x, 5)\} \\ \text{If } x \in H \;&\Rightarrow\; q((x, 5)) = [(x, 5)]_H = \{(x, 4), (x, 5)\} \\ \text{If } x \in K \;&\Rightarrow\; q((x, 5)) = [(x, 5)]_K = \{(x, 5), (x, 6)\} \end{aligned}$$

Equip $\mathscr{D}$ with the quotient topology τ_q induced by q, to obtain the quotient space $(\mathscr{D}, \tau_q)$ induced by q. Then U is open in $\mathscr{D}$ provided $q^{\leftarrow}[U]$ is open Z.

(a) The decomposition space $\mathscr{D}\backslash\{\omega\}$ is upper semicontinuous.
(b) The quotient map $q : Z \to \mathscr{D}$ is a perfect map.
(c) The subset $q[S \times \{r\}]$ is closed in $\mathscr{D}$ for all $r \in \mathbb{N}$.
(d) The space $(\mathscr{D}, \tau_q)$ is Hausdorff. If S is regular than $\mathscr{D}$ is regular.
(e) The sets S and $q[S \times \{r\}]$ are homeomorphic for all $r \in \mathbb{N}$.

Proof of (a): We are given that S is Hausdorff.

(a) We are required to show that $\mathscr{D}\backslash\{\omega\}$ is an upper semicontinuous decomposition of $S \times \mathbb{N}$.

Case 1. Let (a, r) in $S \times \mathbb{N}$ such that $a \notin H \cup K$.

Let U be an open neighborhood of (a, r) in $S \times \{r\} \subseteq S \times \mathbb{N}$. There exists an open V in S such that $V \cap (H \cup K) = \varnothing$ and $(a, r) \in q^{\leftarrow}[V \times \{r\}] \subseteq U$. So U contains a q-saturated open neighborhood of (a, r).

Case 2. Suppose $(h, n) \in Z$ such that $h \in H$ and n is even.

Let W be an open neighborhood of $(h, n) \in S \times \mathbb{N}$. Since H and K are disjoint closed subsets of S, there exists an open neighborhood V of h in S such that $V \cap K = \varnothing$ and $V \times \{n, n+1\} \subseteq W$.

Then $V \times \{n, n+1\} = (V \cap H) \times \{n, n+1\}) \cup (V\backslash H) \times \{n, n+1\}$.

Clearly the subset $(V\backslash H) \times \{n, n+1\}$ is q-saturated.

Since

$$\begin{aligned}
q^{\leftarrow}[q[(V \cap H) \times \{n, n+1\}]] &= q^{\leftarrow}[\{[(x, n)]_H : x \in V \cap H\}] \\
&= q^{\leftarrow}[\{ [(x, n+1)]_H : x \in V \cap H\}] \\
&= \{ \{(x, n), (x, n+1)\} : x \in V \cap H\} \\
&= (V \cap H) \times \{n, n+1\}
\end{aligned}$$

Then $(V \cap H) \times \{n, n+1\}$ is q-saturated.

Then W is a q-saturated open neighborhood of $(h, n) \in Z$ such that $h \in H$ where n is even.

Case 3. Suppose $(k, n) \in Z$ such that $k \in K$ and k is odd. Let W be an open neighborhood of $(k, n) \in S \times \mathbb{N}$. There exists an open neighborhood V of h in S such that $V \cap H = \varnothing$ and $V \times \{n, n+1\} \subseteq W$.

As in case 2, $q^{\leftarrow}[q[V \times \{n, n+1\}]] = V \times \{n, n+1\}]$.

We conclude that $\mathscr{D}\backslash\{\omega\}$ is upper semicontinuous, as required.

Proof of (b): Since the quotient map of an upper semicontinuous decomposition is closed, $q : S \times \mathbb{N} \to \mathscr{D}\backslash\{\omega\}$ is a closed quotient map. Suppose F is closed in $S \times \mathbb{N}$ and ω is a limit point of F in Z. Then $q[F \cup \{\omega\}] = q[F] \cup \{\omega\}$ where F is closed in $\mathscr{D} \setminus \{\omega\}$. Since ω is a limit point of F in Z and q is continuous then $q[F]$ contains its limit point $\{\omega\}$ in $\mathscr{D}$. So $q[F \cup \{\omega\}]$ is closed in $\mathscr{D}$. We conclude that $q : Z \to \mathscr{D}$ is a closed quotient map.

Clearly, for every element d in $\mathscr{D}$, $q^{\leftarrow}(d)$ is compact in Z.

So $q : Z \to \mathscr{D}$ is a perfect function.

Proof of (c): Since $q : Z \to \mathscr{D}$ is a perfect map and $S \times \{r\}$ is closed in Z for all $r \in \mathbb{N}$, then $q[S \times \{r\}]$ is closed for all $r \in \mathbb{N}$.

For example, If n is even,

$$\begin{aligned} q[S \times \{n\}] &= \{q(x,n) : x \notin H \cup K\} \cup \{q(x,n) : x \in H\} \cup \{q(x,n) : x \in K\} \\ &= \{[x,n] : x \notin H \cup K\} \cup \{[x,n]_H : x \in H\} \cup \{[x,n]_K : x \in K\} \\ &= \{\{(x,n)\} : x \notin H \cup K\} \\ &\qquad \cup \{\{(x,n),(x,n+1)\} : x \in H\} \\ &\qquad\qquad \cup \{(x,n-1),(x,n)\} : x \in K\} \end{aligned}$$

If n is odd,

$$\begin{aligned} q[S \times \{n\}] &= \{q(x,n) : x \notin H \cup K\} \cup \{q(x,n) : x \in H\} \cup \{q(x,n) : x \in K\} \\ &= \{[x,n] : x \notin H \cup K\} \cup \{[x,n]_H : x \in H\} \cup \{[x,n]_K : x \in K\} \\ &= \{\{(x,n)\} : x \notin H \cup K\} \\ &\qquad \cup \{\{(x,n-1),(x,n)\} : x \in H\} \\ &\qquad\qquad \cup \{(x,n),(x,n+1)\} : x \in K\} \end{aligned}$$

Proof of (d): Since $q : Z \to \mathscr{D}$ has been shown to be a perfect map, then based on the result proven in the related example on page 628 if S is Hausdorff then $\mathscr{D}$ is Hausdorff and if S is regular then $\mathscr{D}$ is regular.

Proof of (e): Let $f : S \to S \times \{r\}$ (for a fixed $r \in \mathbb{N}$) be defined as $f(x) = (x,r)$. Clearly, f maps S homeomorphically onto $S \times \{r\}$. We know that the function $q : S \times \{r\} \to \{[(x,r)] : x \in S\}$ defined as $q((x,r)) = [x,r]$ is continuous and one-to-one.

Then the function $h = q \circ f$ is continuous and one-to-one. To show that h is a homeomorphism it suffices to show that h is open.

Let U be an open subset of S. See that $q^{\leftarrow}[q[U \times \{r\}]] = q^{\leftarrow}[\{[x,r] : x \in U\}] = U \times \{r\}$. Then $q[U \times \{r\}]$ is open in $\mathscr{D}$. Then h is a homeomorphism.

Theorem E4. *A regular space which is not completely regular.* Let S be a non-normal topological which contains disjoint closed subsets H and K which cannot be separated by disjoint open subsets of S.
Let

$$Z = (S \times \mathbb{N}) \cup \{\omega\}$$

be a set equipped with the topology τ_Z described above.
Let $(\mathscr{D}, \tau_q)$ be the quotient space generated by the quotient map $q : S \to \mathscr{D}$ as described above. Then $(\mathscr{D}, \tau_q)$ is a regular space which is not completely regular.

Proof: We are required to show that $(\mathscr{D}, \tau_q)$ is a regular space which is not completely regular. Suppose there exists a continuous function $f : \mathscr{D} \to [0, 1]$ such that

$$f[q[S \times \{1\}]] = \{1\} \quad \text{and} \quad f[\{\omega\}] = \{0\}$$

We will derive from this assumption some contradiction.

Let

$$\{x_i : i \in \mathbb{N}\backslash\{0\}\}$$

be a strictly increasing sequence in the open interval $(0, 1/2) \subset f[\mathscr{D}]$ which converges to $1/2$ (where $x_1 \neq 0$).
For each $i = 1, 2, 3, \ldots$, we let

$$V_i = f^{\leftarrow}[[0, x_i)] \text{ and } A_i = f^{\leftarrow}[[x_i, x_{i+1})]$$

It represents a collection of subsets of $\mathscr{D}$ where

$$\{\omega\} \in f^{\leftarrow}[[0, x_1)] = V_1 \subset V_2 \subset V_3 \subset \cdots$$

The set $V_1 = f^{\leftarrow}[[0, x_1)]$ is an open neighborhood of $\{\omega\}$. Then $q^{\leftarrow}[V_1]$ is an open neighborhood of ω in Z. It contains a basic open neighborhood of the form

$$B_n = \{\omega\} \cup (S \times \{n + 1, n + 2, n + 3, \ldots\})$$

We can choose an even value for n such that $H \times \{n\} \subseteq q^{\leftarrow}[V_1]$. Hence, for this even n, $q[H \times \{n\}] \subseteq V_1$.

We introduce some notation: Let

$$\begin{aligned} H_n &= q[H \times \{n\}] \\ K_n &= q[K \times \{n\}] \end{aligned}$$

To better illustrate the mechanism behind the following arguments suppose, for example, $n = 6$ so that $H_6 \subseteq V_1 = f^{\leftarrow}[(0, x_1)]$[1]

[1]Note that in what follows $K_6 = K_5$.

Claim #1. We claim that

$$f^{\leftarrow}(x_1) \cap K_5 \neq \varnothing$$

Proof of claim: Since H_6 and K_5 cannot be contained in disjoint open neighborhoods and $H_6 \subseteq V_1 \backslash K_5 \subseteq V_1$, by Lemma E1,

$$H_6 \text{ and } \mathrm{cl}_{\mathscr{D}}(V_1) \;\cap\; K_5$$

cannot be contained in disjoint open subsets.
So there exists $z \in \mathrm{cl}_{\mathscr{D}}[V_1 \backslash K_5] \;\cap\; (\mathrm{cl}_{\mathscr{D}}[V_1] \cap K_5)$.
Since

$$\mathrm{cl}_{\mathscr{D}}[V_1 \backslash K_5] \;\cap\; (\mathrm{cl}_{\mathscr{D}}[V_1] \cap K_5) = \mathrm{cl}_{\mathscr{D}}[V_1 \backslash K_5] \cap K_5$$

and

$$[V_1 \backslash K_5] \cap K_5 = \varnothing$$

then

$$[\,\mathrm{cl}_{\mathscr{D}}(V_1 \backslash K_5)\,] \backslash (V_1 \backslash K_5) \;\cap\; K_5 \neq \varnothing$$

Given that $f^{\leftarrow}[[0, x_1)] \subseteq f^{\leftarrow}[[x_0, x_1]] \;\Rightarrow\; \mathrm{cl}_{\mathscr{D}} f^{\leftarrow}[[0, x_1)] \subseteq f^{\leftarrow}[[x_0, x_1]]$ we can deduce that

$$z \in [\,\mathrm{cl}_{\mathscr{D}}(f^{\leftarrow}[[0, x_1)] \backslash K_5)\,] \setminus (f^{\leftarrow}[[0, x_1)] \backslash K_5) \quad \Rightarrow \quad z \in f^{\leftarrow}(x_1)$$

and since $z \in K_5$ then

$$f^{\leftarrow}(x_1) \cap K_5 \neq \varnothing$$

as claimed.
Also note: Given that $A_1 = f^{\leftarrow}[[x_1, x_2)]$ and $f^{\leftarrow}(x_1) \cap K_5 \neq \varnothing$ then

$$A_1 \cap K_5 \neq \varnothing$$

Step 1. Since H_6 and $K_6 = K_5$ cannot be separated by disjoint open subsets and $H_6 \subseteq V_1 = f^{\leftarrow}[(0, x_1)$, then

$$\mathrm{cl}_{\mathscr{D}} V_1 \cap K_5 \neq \varnothing$$

Claim #2. We claim that the two subsets $\mathrm{cl}_{\mathscr{D}} V_1$ and $A_1 \cap K_5$ cannot be separated by disjoint open subsets.

Proof of claim.

$$\begin{aligned} z \in f^{\leftarrow}[\{x_1\}] \;\cap K_5 \quad &\Rightarrow \quad z \in [\, f^{\leftarrow}[\{x_1\}] \cup f^{\leftarrow}[(x_1, x_2)]\,] \cap K_5 \\ &= \quad z \in f^{\leftarrow}[[x_1, x_2)] \cap K_5 \\ &= \quad z \in A_1 \cap K_5 \end{aligned}$$

Then $z \in \mathrm{cl}_{\mathscr{D}} V_1 \cap A_1 \cap K_5$ where

$$V_1 \cap A_1 \cap K_5 = \varnothing$$

Then the two subsets $\mathrm{cl}_{\mathscr{D}} V_1$ and $A_1 \cap K_5$ cannot be separated by disjoint open subsets, as claimed.

Note that by definition of A_1, the non-empty subset $A_1 \cap K_5 \subseteq V_2$.

Since $A_1 \cap K_5 \subseteq V_2$, then by Lemma E1, the two closed subsets

$$(\mathrm{cl}_{\mathscr{D}} V_2) \cap H_6 \quad \text{and} \quad \mathrm{cl}_{\mathscr{D}} A_1 \cap K_5$$

cannot be separated by disjoint open subsets.
Since $(\mathrm{cl}_{\mathscr{D}} V_2) \cap H_6 = H_6$,

$$H_6 \quad \text{and} \quad \mathrm{cl}_{\mathscr{D}} A_1 \cap K_5$$

cannot be separated by disjoint open subsets.

We repeat he procedure for the two non-empty subsets $A_1 \cap K_5$ and H_5 (where $A_1 \cap K_5 \neq \varnothing$ and $A_1 \cap K_5 \subseteq V_2$). Note that $H_5 = H_4$. As shown in Claim #1,

$$f^{\leftarrow}[\{x_2\}] \cap H_4 \neq \varnothing$$

Step 2. Since $A_1 \cap K_5$ and $H_5 = H_4$ cannot be separated by disjoint open subsets and $A_1 \cap K_5 \subseteq V_2 = f^{\leftarrow}[(0, x_2)$, then

$$\mathrm{cl}_{\mathscr{D}} f^{\leftarrow}[[0, x_2)] \cap H_4 = \mathrm{cl}_{\mathscr{D}} V_2 \cap H_4 \neq \varnothing$$

As in Claim #1,

$$f^{\leftarrow}(x_2) \cap H_4 \neq \varnothing$$

As in Claim #2, we prove that $\mathrm{cl}_{\mathscr{D}} V_2$ and $A_2 \cap H_4$ cannot be separated by disjoint open subsets.

$$\begin{aligned} z \in f^{\leftarrow}[\{x_2\}] \ \cap H_4 \quad &\Rightarrow \quad z \in [\, f^{\leftarrow}[\{x_2\}] \cup f^{\leftarrow}[(x_2, x_3)]\,] \cap H_4 \\ &= \quad z \in f^{\leftarrow}[[x_2, x_3)] \cap H_4 \\ &= \quad z \in A_2 \cap H_4 \end{aligned}$$

Then $z \in \mathrm{cl}_{\mathscr{D}} V_2 \cap (A_1 \cap K_5)$ where

$$V_2 \cap (A_2 \cap H_4) = f^{\leftarrow}[[0, x_2)] \cap (f^{\leftarrow}[[x_2, x_3)] \cap H_4) = \varnothing$$

Then the two subsets $\mathrm{cl}_{\mathscr{D}} V_2$ and $A_2 \cap H_4$ cannot be separated by disjoint open subsets, as claimed.

Note that $A_1 \cap K_5 \neq \varnothing$ and $A_1 \cap K_5 \subseteq V_2$.
Since $A_1 \cap K_5 \subseteq V_2$, then by Lemma E1, the two subsets

$$(\mathrm{cl}_{\mathscr{D}} V_2) \cap H_6 \quad \text{and} \quad A_1 \cap K_5$$

cannot be separated by disjoint open subsets.
Since $A_2 \cap H_4 \neq \varnothing$ and $A_2 \cap H_4 \subseteq V_3$, then by Lemma E1, the two subsets

$$V_3 \cap (A_1 \cap K_5) = A_1 \cap K_5 \quad \text{and} \quad A_2 \cap H_4$$

cannot be separated by disjoint open subsets.
This completes the Step 2.

We summarize the final steps to arrive at a contradiction.

Step 1. Given $H_6 \subseteq V_1$, "The sets H_6 and K_5 cannot be separated by disjoint open subsets where " implies "The sets H_6 and $A_1 \cap K_5$. cannot be separated by disjoint open subsets, where $A_1 \cap K_5 \neq \varnothing$ and $A_1 \cap K_5 \subseteq V_2$".

Step 2. Given $A_1 \cap K_5 \subseteq V_2$,"The sets $A_1 \cap K_5$ and H_6 cannot be separated by disjoint open subsets" implies "The sets $A_1 \cap K_5$ and $A_2 \cap H_4$. cannot be separated by disjoint open subsets, where $A_2 \cap H_4 \neq \varnothing$ and $A_2 \cap H_4 \subseteq V_3$".

Step 3. Given $A_2 \cap H_4 \subseteq V_3$,"The sets $A_1 \cap K_5$ and $A_2 \cap H_4$ cannot be separated by disjoint open subsets" implies "The sets $A_2 \cap H_4$ and $A_3 \cap K_3$, where $A_3 \cap K_3 \neq \varnothing$ and $A_3 \cap K_3 \subseteq V_4$".

We work our way down to

Step 5. Given $A_4 \cap H_2 \subseteq V_5$,"The sets $A_4 \cap H_2$ and $A_3 \cap K_3$ cannot be separated by disjoint open subsets" implies "The sets $A_4 \cap H_2$ and $A_5 \cap K_1$, where $A_5 \cap K_1 \neq \varnothing$ and $A_5 \cap K_1 \subseteq V_6$".

From the result of Step 5, we derive a contradiction.

Claim #3. We claim that $f[q[S \times \{1\}]] \cap [0, 1/2] \neq \varnothing$.
Proof of claim: See that $K_1 \subseteq q[S \times \{1\}]$, that $A_5 \cap K_1 \neq \varnothing$ and $A_5 \cap K_1 \subseteq V_6 = f^{\leftarrow}[[0, x_6)]$.

Then

$$f[K_1 \cap q[S \times \{1\}]] \subseteq [0, x_6) \subseteq [0, 1/2]$$

Then $f[q[S \times \{1\}]] \cap [0, 1/2] \neq \varnothing$, as claimed.
Since $f[q[S \times \{1\}]] = \{0, \}$, we have a contradiction.
The source of our contradiction is our initial assumption that there exists a continuous function $f : \mathscr{D} \to [0, 1]$ such that

$$f[q[S \times \{1\}]] = \{1\} \text{ and } f[\{\omega\}] = \{0\}$$

So $(\mathscr{D}, \tau_q)$ is a regular space which is not completely regular.

We are done.