Example 14. *The ψ-space is not normal.* In an example found on page 110 we defined the space

$$(\psi, \tau)$$

where $\psi = \mathbb{N} \cup \mathscr{M}$ and where $\mathscr{M}$ is a collection of infinite subsets of $\mathbb{N}$ in which every pair has finite intersection and is such that $\mathscr{M}$ is maximal with respect to this property. We showed that the set

$$\{\{n\} : n \in \mathbb{N}\} \cup \{\{D\} \cup D\backslash F : D \in \mathscr{M}, F \text{ is a finite subset of } D\}$$

forms a base of a topology τ on ψ.

Show that the space (ψ, τ) is not a normal space.

Solution: In Theorem 10.9, we showed that a topological space S is normal if and only if every closed subset of S is C^*-embedded in S. To we show that (ψ, τ) is not normal we will exhibit a closed subset of ψ which is not C^*-embedded in ψ.

We are given the space (ψ, τ) where $\psi = \mathbb{N} \cup \mathscr{M}$. The subset $\mathscr{M}$ is easily seen to be closed in ψ. We showed in the example on page 184 that $\mathscr{M}$ is an uncountably large set. Let $\mathscr{N} = \{\{D_i\} : i \in \mathbb{N}\}$ be a countably infinite subset of $\mathscr{M}$. Then $\mathscr{M}\backslash\mathscr{N}$ is uncountable.

For each $i \in \mathbb{N}$, $\{D_i\} \cup D_i\backslash F_i$ is a basic open neighborhood of D_i. Since

$$\{D_i\} \ = \ [\{D_i\} \cup (D_i\backslash F_i)] \ \cap \ \mathscr{M}$$

$\{D_i\}$ is open in the subspace, $\mathscr{M}$. Then $\mathscr{N}$ is an open subset of the subspace $\mathscr{M}$. Similarly, $\mathscr{M}\backslash\mathscr{N}$ is open in $\mathscr{M}$. Then $\mathscr{N}$ and $\mathscr{M}\backslash\mathscr{N}$ are disjoint clopen subsets of the space $\mathscr{M}$. The subsets $\mathscr{N}$ and $\mathscr{M}\backslash\mathscr{N}$ are then completely separated in $\mathscr{M}$.

The Urysohn's extension theorem (stated on page 250) states that a subset T of S is C^*-embedded in S if and only if subsets which are completely separated in T are also completely separated in S. To show that ψ is not normal, it then suffices to show that $\mathscr{N}$ and $\mathscr{M}\backslash\mathscr{N}$ are *not* completely separated in ψ.

Let U be an open subset of ψ such that $U \cap \mathscr{M} = \mathscr{M}\backslash\mathscr{N}$ and let $D \in \mathscr{N}$. If we can show that $D \in \text{cl}_\psi U \cap \mathscr{N}$ then $\mathscr{N}$ and $\mathscr{M}\backslash\mathscr{N}$ are not completely separated in ψ.

Let $D = \{x_i : i = 0, 1, 2, 3, \ldots\}$ and, for $n \in \mathbb{N}$, $F_n = \{x_0, x_1, x_2, \ldots, x_n\}$. There exists $D_0 \in \mathscr{M}\backslash\mathscr{N}$ such that $D_0 \cap D = F_0$. Then

$$(\{D_0\} \cup D_0\backslash F_0) \cap (\{D\} \cup D\backslash F_0) \ = \ (D_0 \cap D)\backslash F_0 \ = \ \varnothing$$

Let $F_1 = \{x_1\} \cup F_0$.

There exists $D_1 \in (\mathscr{M} \backslash \mathscr{N}) \backslash \{D_0\}$ such that $D_1 \cap D = F_1 = \{x_1\} \cup F_0$ (where $\{D\} \cup D \backslash F_1 \subseteq \{D\} \cup D \backslash F_0$). Then

$$(\{D_1\} \cup D_1 \backslash F_1) \cap (\{D\} \cup D \backslash F_1) \;=\; (D_1 \cap D) \backslash F_1 \;=\; \varnothing$$

If $F_2 = \{x_2\} \cup F_1$ there exists $D_2 \in (\mathscr{M} \backslash \mathscr{N}) \backslash \{D_0, D_1\}$ such that $D_2 \cap D = F_2$. Then

$$\{D_2\} \cup D_2/F_2] \;\cap\; [\{D\} \cup D/F_2] \;=\; (D_2 \cap D) \backslash F_2 \;=\; \varnothing$$

More generally, for $F_n \subseteq D$ and $F_{n+1} = \{x_{n+1}\} \cup F_n$, there exists $D_{n+1} \in (\mathscr{M} \backslash \mathscr{N}) \backslash \{D_0, D_1, \ldots, D_n\}$ such that $D_{n+1} \cap D = F_{n+1}$. This gives

$$(\{D_{n+1}\} \cup D_{n+1}/F_{n+1}) \;\cap\; (\{D\} \cup D/F_{n+1}) \;=\; (D_{n+1} \cap D) \backslash F_{n+1} = \varnothing$$

We thus construct the infinite sequences of sets

$$\{\{D_n\} \cup D_n \backslash F_n : n \in \mathbb{N}\} \text{ and } \{\{D\} \cup D \backslash F_n : n \in \mathbb{N}\}$$

in ψ. For each $n \in \mathbb{N}$, let

$$\begin{aligned} U_n &= \cup\{\{D\} \cup D \backslash F_i : i > n\} \\ V_n &= \cup\{\,(\{D_i\} \cup D_i \backslash F_i\,) \;\cap\; U : i > n\} \\ W_n &= U_n \cup V_n \end{aligned}$$

where $U_{n+1} \subset U_n$ and $V_{n+1} \subset V_n$. See that for each n, W_n is a ψ-open neighborhood of D (since $D \in U_n = \cup\{\{D\} \cup D \backslash F_i : i > n\}$ for each n).

Then

$$\begin{aligned} \cap\{W_n : n \in \mathbb{N}\} &= \cap\{U_n \cup V_n : n \in \mathbb{N}\} \\ &= (\cap\{U_n : n \in \mathbb{N}\}) \;\cup\; (\cap\{V_n : n \in \mathbb{N}\}) \\ &= \{D\} \;\cup\; (\cap\{V_n : n \in \mathbb{N}\}) \\ &= \{D\} \;\cup\; \varnothing \end{aligned}$$

See that $W_n \cap U \neq \varnothing$ for all n.

Then every neighborhood W_n of D intersects U. So $D \in \mathrm{cl}_\psi U \cap \mathscr{N}$.

So $\mathscr{M} \backslash \mathscr{N}$ and $\mathscr{N}$ cannot be completely separated in ψ.

We conclude that the closed subset $\mathscr{M}$ is not C^*-embedded in ψ. Then ψ cannot be normal.