

Let

$$\begin{aligned} V_1 &= \cup_{i=1}^{\infty} \{U_i\} \\ V_2 &= \cup_{i=2}^{\infty} \{U_i\} \\ V_3 &= \cup_{i=3}^{\infty} \{U_i\} \\ &\vdots \\ V_n &= \cup_{i=n}^{\infty} \{U_i\} \\ &\vdots \end{aligned}$$

where $\{k_1, k_2, k_3, \dots, k_n\} \cap V_{n+1} = \emptyset$ and $\{k_{n+1}, k_{n+2}, \dots\} \subseteq V_{n+1}$.

Then $\cap \{V_n : n = 1, 2, 3, \dots\} = \emptyset$. We then have an infinite subset of \mathbb{N}

$$A = \{k_1, k_2, k_3, \dots\}$$

such that $A \cap D$ is finite for all $D \in \mathcal{M}$. This contradicts the maximality of \mathcal{M} . We can then conclude that there can be no unbounded continuous real-valued function on ψ . So ψ is pseudocompact.

We know that a space is countably compact if and only if every sequence T has an accumulation point p (that is, every neighborhood of p intersects every tail-end of the sequence T). To show that ψ is not countably compact, it suffices to show that it contains a sequence of distinct points with no accumulation point. Consider a sequence of points in \mathcal{M} . We know that \mathcal{M} is a closed subset of ψ and is a discrete subspace. If T is sequence in \mathcal{M} no point of ψ can be an accumulation point of T . So ψ is not countably compact.

Remark: The above example shows that *pseudocompactness is not a hereditary property*, since ψ has been shown to be pseudocompact while the subset \mathcal{M} is an uncountable discrete subset of ψ and so is not pseudocompact. . In the next chapter we show that the ψ -space is completely regular, from which it will follow that it is feebly compact.

17.6 Pseudocompact but not feebly compact.

In the following example we present an example of a topological Hausdorff space which is *not feebly compact* but is pseudocompact. We will first need the interesting result presented in the following proposition. Its proof is deferred to an Appendix D of the text.

Proposition 17.12 Let $S = [0, 1]$ (the closed unit interval) equipped with the usual topology. Then S can be partitioned into infinitely many dense subsets of S (with respect to the usual topology).

Proof: The proof is deferred to Appendix D of the text.

Example 4. A non-feebly-compact pseudocompact space. Let $S = [0, 1]$ (the closed unit interval). Let

$$\mathcal{B} = \{U_i : i = 1, 2, 3, \dots\}$$

denote a partition of S where each U_i is a dense subset of S (with respect to the usual topology). That is, \mathcal{B} is a collection of *pairwise disjoint* (not necessarily open) subsets of S which covers S .

We will call on this particular fact to construct a topology on $S = [0, 1]$ as follows:

Suppose τ denotes the usual topology on S . We know that (S, τ) is Hausdorff.

For each $i \in \mathbb{N} \setminus \{0\}$, from $\{U_1, U_2, U_3, \dots\}$ we define

$$V_i = U_{2i-1} \cup U_{2i} \cup U_{2i+1}$$

So $\{V_i : i = 1, 2, 3, \dots\}$ covers $S = [0, 1]$. That is, $S \setminus \cup\{V_i : i = 1, 2, 3, \dots\} = \emptyset$.⁵

Define the collection γ of subsets of S as,

$$\gamma = \{U_{2i-1} : i \in \mathbb{N} \setminus \{0\}\} \cup \{V_i : i \in \mathbb{N} \setminus \{0\}\}$$
⁶

View the subset

$$\mathcal{S} = \tau \cup \gamma$$

of $\mathcal{P}(S)$ as a subbase which will generate a topology, say σ , on S . By definition, $\tau \subseteq \sigma$ so every element of τ is an element of the subbase which generates (S, σ) . So

$$\tau \subseteq \sigma$$

Since (S, τ) is Hausdorff, then the space, (S, σ) , is also Hausdorff. Also, in (S, σ) each V_i is open; hence, for each i , $S \setminus V_i$ is closed in S .

⁵ $\{V_i : i = 1, 2, 3, \dots\} = \{U_1 \cup U_2 \cup U_3, U_3 \cup U_4 \cup U_5, U_5 \cup U_6 \cup U_7, U_7 \cup U_8 \cup U_9, \dots\}$

⁶ $\gamma = \{U_1, U_3, U_5, \dots\} \cup \{U_1 \cup U_2 \cup U_3, U_3 \cup U_4 \cup U_5, U_5 \cup U_6 \cup U_7, \dots\}$

(a) Show that (S, σ) , just described, is *not* feebly compact.

Solution: We will set up the proof so as to apply part (3) of Theorem 17.8: “Decreasing $\{U_i : i \in \mathbb{N}\}$ of open subsets of $S \Rightarrow \bigcap \{\text{cl}_S U_i : i \in \mathbb{N}\} \neq \emptyset$ ”.

Recall that $S = \bigcup \{U_i : i = 1, 2, 3, \dots\}$. Let

$$\begin{aligned} A_1 &= \{U_3 \cup U_5 \cup U_7 \cup U_9 \cup \dots\} \\ A_2 &= \{U_5 \cup U_7 \cup U_9 \cup U_{11} \cup \dots\} \subseteq S \setminus V_1 \\ A_3 &= \{U_7 \cup U_9 \cup U_{11} \cup U_{13} \cup \dots\} \subseteq S \setminus (V_1 \cup V_2) \\ &\vdots \\ A_n &= \{U_{2n+1} \cup U_{2n+3} \cup U_{2n+5} \cup \dots\} \subseteq S \setminus (V_1 \cup V_2 \cup \dots \cup V_{n-1}) \\ &\vdots \end{aligned}$$

Then, for $\{A_n : n = 1, 2, 3, \dots\}$,

$$A_{n+1} \subseteq A_n \text{ and } \text{cl}_S A_{n+1} \subseteq \text{cl}_S A_n$$

So $\{A_n : n = 1, 2, 3, \dots\}$ is a nested collection of open subsets of (S, σ) . To show that (S, σ) is not feebly compact, it suffices to show that $\bigcap \{\text{cl}_S A_n : n = 1, 2, 3, \dots\} = \emptyset$.

Verify that

$$\begin{aligned} \text{cl}_S A_2 &= \text{cl}_S \{U_5 \cup U_7 \cup U_9 \cup \dots\} \subseteq S \setminus (U_1 \cup U_2 \cup U_3) = S \setminus V_1 \\ \text{cl}_S A_3 &= \text{cl}_S \{U_7 \cup U_9 \cup U_{11} \cup \dots\} \subseteq S \setminus (U_1 \cup \dots \cup U_5) = S \setminus (V_1 \cup V_2) \\ &\vdots \\ \text{cl}_S A_n &\subseteq S \setminus (V_1 \cup V_2 \cup \dots \cup V_{n-1}) \\ \text{cl}_S A_{n+1} &\subseteq S \setminus (V_1 \cup V_2 \cup \dots \cup V_n) \\ &\vdots \end{aligned}$$

Then

$$\begin{aligned} \bigcap \{\text{cl}_S A_n : n = 1, 2, 3, \dots\} &\subseteq \bigcap \{S \setminus (V_1 \cup \dots \cup V_{n-1}) : n = 1, 2, 3, \dots\} \\ &= S \setminus \{V_n : n = 1, 2, 3, \dots\} \\ &= S \setminus S \\ &= \emptyset \end{aligned}$$

By Theorem 17.8, part 3), since each A_i is open in (S, σ) , (S, σ) is not feebly compact, as required.

(b) Show that the non-feebly-compact topological space, (S, σ) is not regular.

Solution: Let $a \in S = [0, 1]$. Suppose, for example, that $a \in U_3 \in \mathcal{B}$. Since U_3 is a σ -subbase element it is σ -open. The set U_4 is easily verified to be the complement of a σ -open set and so is σ -closed. Since \mathcal{B} partitions S , then $a \notin U_4$. To show that S is not σ -regular, it suffices to show that a and the closed subset, U_4 , cannot be separated by disjoint σ -open subsets.

Let (r, s) be a τ -basic open interval (in $[0, 1]$) containing a . Then $(r, s) \cap U_3$ is a σ -basic open set containing a .

Claim: We claim that every σ -open neighborhood of U_4 intersects $(r, s) \cap U_3$.

Proof of claim: Since U_4 is dense in S , the open interval (r, s) containing a must intersect U_4 . Say $q \in (r, s) \cap U_4$ and (w, z) is an open interval containing q . The open interval $(r, s) \cap (w, z)$ contains q . Since U_3 is dense in S , then $(r, s) \cap (w, z) \cap U_3 \neq \emptyset$.

See that

$$\begin{aligned} (r, s) \cap (w, z) \cap U_3 &= [(r, s) \cap (w, z)] \cap [U_3 \cup U_4 \cup U_5] \cap U_3 \\ &= [(w, z) \cap U_2] \cap [(r, s) \cap U_3] \\ &\neq \emptyset \end{aligned}$$

So every σ -open neighborhood of U_4 will intersect the σ -open neighborhood of a .

So (S, σ) is not regular.

(c) Let $f : S \rightarrow \mathbb{R}$ be any continuous real-valued function on S . Show that, if $(c, d) \cap f^{-}[(a, b)]$ is non-empty, then $(c, d) \cap f^{-}[(a, b)] \cap U_k$ is non-empty for all U_k in \mathcal{B} .

Solution: Since f is continuous on S_σ , then $f^{-}[(a, b)]$ is a σ -open subset of S_σ . We are given that $(c, d) \cap f^{-}[(a, b)] \neq \emptyset$. Suppose $p \in (c, d) \cap f^{-}[(a, b)] \cap U_k$, for some k .

We claim that neither $(c, d) \cap f^{-}[(a, b)] \cap U_{k-1}$ nor $(c, d) \cap f^{-}[(a, b)] \cap U_{k+1}$ can be empty.

Proof of claim: The number k is either even or odd. We consider these two cases separately.

Case 1. If k is even. We then have

$$p \in U_{k-1} \cup U_k \cup U_{k+1} \subseteq f^{\leftarrow}[(a, b)]$$

Then, since $U_{k-1} \cup U_k \cup U_{k+1}$ is the smallest σ -open set which contains U_k ,

$$\begin{aligned} (c, d) \cap U_{k-1} &= (c, d) \cap [U_{k-1} \cup U_k \cup U_{k+1}] \cap U_{k-1} \\ &\subseteq (c, d) \cap f^{\leftarrow}[(a, b)] \cap U_{k-1} \end{aligned}$$

Since $(c, d) \cap U_{k-1} \neq \emptyset$, then

$$(c, d) \cap f^{\leftarrow}[(a, b)] \cap U_{k-1} \neq \emptyset$$

Also,

$$\begin{aligned} (c, d) \cap U_{k+1} &= (c, d) \cap [U_{k-1} \cup U_k \cup U_{k+1}] \cap U_{k+1} \\ &\subseteq (c, d) \cap f^{\leftarrow}[(a, b)] \cap U_{k+1} \end{aligned}$$

Since, $(c, d) \cap U_{k+1} \neq \emptyset$, then

$$(c, d) \cap f^{\leftarrow}[(a, b)] \cap U_{k+1} \neq \emptyset$$

Case 2. If k is odd, with $p \in (c, d) \cap f^{\leftarrow}[(a, b)] \cap U_k$. Then U_k is a σ -open neighborhood of p . We have $U_{k-1} \cup U_k \cup U_{k+1}$ is the σ -closure of U_k . Since $f(p) \in (a, b)$ there exists r, s such that $f(p) \in [r, s] \subseteq (a, b)$. Then

$$p \in \text{cl}_{S_\sigma} U_k = U_{k-1} \cup U_k \cup U_{k+1} \subseteq f^{\leftarrow}[r, s] \subseteq f^{\leftarrow}[(a, b)]$$

So, for the case where k is odd, we again obtain,

$$(c, d) \cap f^{\leftarrow}[(a, b)] \cap U_{k-1} \neq \emptyset \neq (c, d) \cap f^{\leftarrow}[(a, b)] \cap U_{k+1} \neq \emptyset$$

We conclude that for any $k \in \mathbb{N}$ and non-empty U_k

$$(c, d) \cap f^{\leftarrow}[(a, b)] \cap U_{k-1} \neq \emptyset \neq (c, d) \cap f^{\leftarrow}[(a, b)] \cap U_{k+1} \neq \emptyset$$

We conclude, by induction that, if $(c, d) \cap f^{\leftarrow}[(a, b)]$ intersects one U_k for some k in \mathbb{N} , then $(c, d) \cap f^{\leftarrow}[(a, b)]$ intersects all U_i 's.

(d) Show that, for $U_k \in \mathcal{B}$, the subspace topology on U_k inherited from σ is the same as the subspace topology inherited from τ .

Solution: We show that the subspace topology on U_k inherited from σ is the same as the subspace topology inherited from τ .

Suppose V is an open proper subset of U_k with respect to the subspace topology inherited from τ . Since the U_i 's are pairwise disjoint then V is a union of elements from τ as a subset of σ .

On the other hand, suppose V is an open proper subset of U_k with respect to the subspace topology inherited from σ . Since the U_i 's are pairwise disjoint the only subbase element that can intersect U_k are the elements from $\tau \subset \sigma$.

(e) Show that the space S_σ is pseudocompact.

Solution: To show that S_σ is pseudocompact, we will use the standard approach by supposing $f \in C(S_\sigma)$ is unbounded and then derive a contradiction.

Suppose $g \in C(S)$. Let $f = |g|$. We can then construct an unbounded sequence, $\{y_n : n \in \mathbb{N}\}$ in $f[S]$, where $y_{n+1} \geq y_n + 1$. Then, for each n , $f^{-1}[(y_n - \frac{1}{3}, y_n + \frac{1}{3})]$ is a σ -open subset of S . Let

$$\mathcal{U} = \{f^{-1}[(y_n - \frac{1}{3}, y_n + \frac{1}{3})] : n \in \mathbb{N}\}$$

Then \mathcal{U} is an infinite collection of pairwise disjoint σ -open subsets of S_σ . For each n , choose $p_n \in f^{-1}[(y_n - \frac{1}{3}, y_n + \frac{1}{3})] \cap S_\sigma$, to form a sequence,

$$D = \{p_n : n \in \mathbb{N}\}$$

of distinct points in S_σ . Then $\lim_{n \rightarrow \infty} f(p_n) = \infty$.

Suppose p is a limit point of the set D with respect to the usual topology, τ . Suppose $p \in U_k$. For each $i \neq 0$, there exists

$$p_{n(i)} \in (p - 1/i, p + 1/i) \cap D$$

which forms a subsequence, $E = \{p_{n(i)} : i \in \mathbb{N}\}$, of D which converges to p with respect to τ .

For a given i , suppose $p_{n(i)} \in U_i$. Then

$$p_{n(i)} \in (p - 1/i, p + 1/i) \cap f^{-1}[(y_{n(i)} - 1/3, y_{n(i)} + 1/3)] \cap U_i$$

Then, by part (c), since $(p - 1/i, p + 1/i) \cap f^{-1}[(y_{n(i)} - 1/3, y_{n(i)} + 1/3)] \neq \emptyset$,

$$(p - 1/i, p + 1/i) \cap f^{-1}[(y_{n(i)} - 1/3, y_{n(i)} + 1/3)] \cap U_k \neq \emptyset$$

(where p was stated as belonging to U_k).

We can then choose, for each i ,

$$q_i \in (p - 1/i, p + 1/i) \cap f^{-1}[(y_{n(i)} - 1/3, y_{n(i)} + 1/3)] \cap U_k$$

We can, in this way, construct a sequence $\{q_i : i \in \mathbb{N}\}$ in U_k where $f(q_i) \in (y_{n(i)} - 1/3, y_{n(i)} + 1/3)$. Then $\lim_{i \rightarrow \infty} f(q_i) = \infty$.

The subspace topology on U_k inherited from S_σ is the same as the subspace topology inherited from τ . So $\lim_{i \rightarrow \infty} q_i$ must equal p . Since f is continuous, and $\lim_{i \rightarrow \infty} f(q_i) = f(p)$. This contradicts $\lim_{i \rightarrow \infty} f(q_i) = \infty$.

The source of our contradiction is our supposition that S_σ is not pseudocompact. So S_σ is pseudocompact.

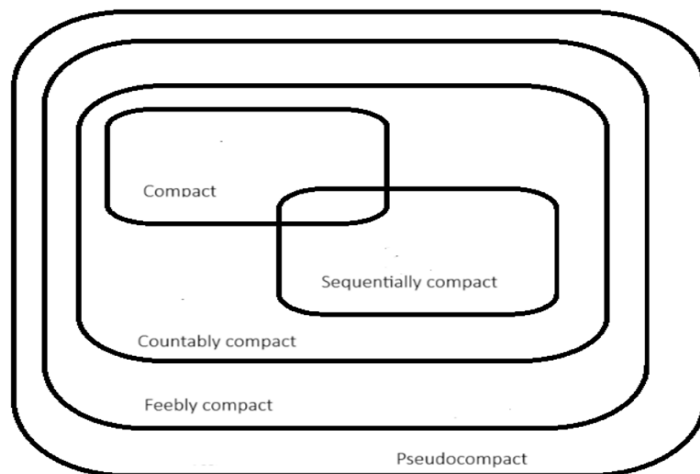


Figure 4: Ranking the compact relatives

Concepts review.

1. Define a sequentially compact space.
2. If S is a second countable or metrizable topological space which satisfies the sequentially compact property, name two other properties which are equivalent properties in such spaces.