Example 6. Is there a function $f : \mathbb{R} \to \mathbb{R}$ which is continuous precisely on $\mathbb{Q}$ (and discontinuous on $\mathbb{R}\backslash\mathbb{Q}$)?

Solution: In the previous example, we showed that, given a function $f : \mathbb{R} \to \mathbb{R}$, the largest subset of $\mathbb{R}$ on which f is continuous is a G_δ set. So, if f is continuous on $\mathbb{Q}$ (and discontinuous elsewhere) the set $\mathbb{Q}$ is a G_δ. We showed in Proposition 34.8 that $\mathbb{Q}$ is not a G_δ. Contradiction! We must then conclude that there can be no real-valued function on $\mathbb{R}$ which is continuous only on $\mathbb{Q}$ (and discontinuous elsewhere).

34.4 The Luzin space.

Assuming the Continuum hypothesis and invoking the Baire category theorem we construct an uncountable subset of $\mathbb{R}$ which intersects every nowhere dense subsets of $\mathbb{R}$ in a countable subset. Such a subspace of $\mathbb{R}$ is called a *Luzin space*. We begin by formally defining it.

Definition 34.10 An uncountable subspace of $\mathbb{R}$ which intersects every closed nowhere dense subset of $\mathbb{R}$ in a countable set is referred to as a *Luzin space* (or *Lusin space*).[1]

A method for constructing a Luzin space is illustrated in the following example.

Example 7.

(a) Show that we can enumerate the collection of all closed nowhere dense subsets of $\mathbb{R}$ as $\mathscr{K} = \{K(\alpha) : \alpha < \omega_1\}$ (CH)

(b) Show that, if we assume the Continuum Hypothesis, there is an uncountably large subset T of $\mathbb{R}$ such that the intersection of T with any closed nowhere subset of S is countable.

Solution: Let $\mathscr{K}$ denote the set of all closed nowhere dense subsets of $\mathbb{R}$.

[1] Nikolai Nikolayevich Luzin (also spelled Lusin) (9 December 1883 - 28 February 1950) was a Soviet and Russian mathematician known for his work in descriptive set theory and aspects of mathematical analysis with strong connections to point-set topology. Pavel Urysohn, Pavel Alexandrov and Andrey Kolmogorov were amongst his brightest doctoral students.

Part (a): Recall that the space $\mathbb{R}$ is second countable, so $\mathbb{R}$ has a countable base, $\mathscr{B}$, for open sets (see example on page 98). To each element F of $\mathscr{K}$, we can associate a dense open subset, $\mathbb{R}\setminus F$, of $\mathbb{R}$, where $\mathbb{R}\setminus F$ is the union of a countable subset of $\mathscr{B}$. Note that $|\mathscr{P}(\mathscr{B})| = 2^{|\mathscr{B}|} = 2^{\aleph_0} = c$.

The Continuum hypothesis (CH) states that $|2^{\aleph_0}|$ is the least uncountable cardinal number greater than $\aleph_0$. That is, there is no uncountable set S such that $\aleph_0 < |S| < |2^{\aleph_0}|$. The Continuum hypothesis is usually stated in the form, $2^{\aleph_0} = \aleph_1$. It has been shown that the Continuum hypothesis cannot be disproved within the confines of the standard axioms of mathematics, ZFC. Nor can the hypothesis be proven from those same ZFC axioms.

We invoke CH to allow us to enumerate the elements of the set, $\mathscr{K}$, of all closed nowhere dense subsets of $\mathbb{R}$ as

$$\mathscr{K} = \{K(\alpha) : \alpha < \omega_1\}$$

where ω_1 is the first uncountable ordinal, the set of all countable ordinals.

Part (b): Recall that *a space S is Baire if the countable union of closed nowhere dense subsets of S with empty interior has empty intersection.* Equivalently, if the countable intersection of dense open subsets of S is dense in S. By the Baire Category theorem, every complete metric space is Baire. So $\mathbb{R}$ is Baire.

Let $\mathscr{K} = \{K(\alpha) : \alpha < \omega_1\}$ denote the enumerated set of all closed nowhere dense subsets of $\mathbb{R}$ as described in part (a).

By the Baire category theorem, for each $\alpha < \omega_1$, $\cup\{K(\beta) : \beta < \alpha\}$ is nowhere dense in $\mathbb{R}$. Equivalently, $\cap\{\mathbb{R}\setminus K(\beta) : \beta < \alpha\}$ is a dense subset of $\mathbb{R}$.

This allows us to inductively construct an uncountable set of *distinct* points

$$T = \{x_\alpha : \alpha < \omega_1\}$$

by choosing the x_α's as follows:

$$
\begin{aligned}
x_1 &\notin K(0) \Leftrightarrow x_1 \in \mathbb{R}\backslash K(0)\\
x_2 &\notin K(1) \ \Leftrightarrow \ x_2 \in \mathbb{R}\backslash K(1); \ \ x_2 \neq x_1\\
x_3 &\notin K(1) \cup K(2)\\
&\qquad\quad \Leftrightarrow x_3 \in \mathbb{R}\backslash K(1) \cap \mathbb{R}\backslash K(2); \ \ x_3 \notin \{x_1, x_2\}\\
x_4 &\notin K(1) \cup K(2) \cup K(3)\\
&\qquad\quad \Leftrightarrow x_4 \in \mathbb{R}\backslash K(1) \cap \mathbb{R}\backslash K(2) \cap \mathbb{R}\backslash K(3); \ \ x_4 \notin \{x_1, x_2, x_3\}\\
&\vdots\\
x_\alpha &\notin \cup\{K(\beta) : \beta < \alpha\}\\
&\qquad\quad \Leftrightarrow x_\alpha \in \cap\{\mathbb{R}\backslash K(\beta) : \beta < \alpha\}; \ \ x_\alpha \notin \{x_\beta : \beta < \alpha\}
\end{aligned}
$$

Given the set $T = \{x_\alpha : \alpha < \omega_1\}$ thus constructed we claim that $T \cap K(\gamma)$ is countable for each $\gamma < \omega_1$.

Proof of claim: Let $\gamma < \omega_1$.

Let $x_\mu \in T \cap K(\gamma)$. Since $x_\mu \notin \cup\{K(\beta) : \beta < \mu\}$ and $x_\mu \in K(\gamma)$ then γ cannot be less than μ. So $\gamma \geq \mu$. So the cardinality of $T \cap K(\gamma)$ cannot be more than γ. Since γ is countable then $T \cap K(\gamma)$ is countable, as claimed.

Thus, T viewed as a subspace of $\mathbb{R}$ is a Luzin space.

Example 8. Construct a Luzin space which is is dense in $\mathbb{R}$.

Solution: As before, $\mathscr{K} = \{K(\alpha) : \alpha < \omega_1\}$ is the enumerated set of all closed nowhere dense subsets of $\mathbb{R}$.

We are required to construct a Luzin set T with satisfies the specific property: The set T is dense in $\mathbb{R}$. This must depend on how we choose the x_α's to form the Luzin set T.

We inductively construct the set $T = \{x_\alpha : \alpha < \omega_1\}$ as follows:

Let $K(0) = \varnothing$.

Choose $x_1 \notin K(0)$, $x_1 \in K(\delta)$, where $\delta =$ the least ordinal ≥ 1 such that $K(\delta) = \{j\}$ for some irrational j.

Choose $x_2 \notin K(1)$, $x_2 \in K(\delta)$, where $\delta =$ the least ordinal ≥ 2 such that $K(\delta) = \{j\}$ for some irrational $j \notin \{x_1\}$.

Choose $x_3 \notin K(1) \cup K(2)$, $x_3 \in K(\delta)$, where $\delta =$ the least ordinal ≥ 3 such that $K(\delta) = \{j\}$ for some irrational $j \notin \{x_1, x_2\}$.

More generally, for each $\alpha < \omega_1$, choose x_α as the only number which simultaneously satisfies the two conditions:

1) $x_\alpha \notin \cup\{K(\beta) : \beta < \alpha\}$
2) $x_\alpha \in K(\delta)$, where $\delta =$ the least ordinal $\geq \alpha$ such that $K(\delta) = \{j\}$ for some irrational $j \notin \{x_\beta : \beta < \alpha\}$

The Baire category theorem guarantees that it is possible to choose the x_α's in this way. By definition, $T = \{x_\beta : \beta < \omega_1\}$ is a Luzin set. The elements of T are of the form,

$$\{j \in \mathbb{J} : \{j\} = K(\gamma) \text{ for some } \gamma, \text{ such that } j \notin \cup\{K(\beta) : \beta < \gamma\}$$

We claim that T, thus constructed, is dense in $\mathbb{R}$.

Proof of claim. It suffices to show that T is dense in $\mathbb{J}$.

Suppose $j \in \mathbb{J}$, $j \notin T$.

Then $j \in K(\gamma) = \{j\}$ for some $\gamma < \omega_1$. If $j \notin \cup\{K(\beta) : \beta < \gamma\}$ then j would be the element x_γ of T. Since it is hypothesized that $j \notin T$, then $j \in \cup\{K(\beta) : \beta < \gamma\}$.

Let $B(j)$ be an open neighborhood of j. Since $K(\gamma)$ is nowhere dense $B(j)\backslash K(\gamma)$ contains an open interval and hence contains uncountably many irrationals. So, for any $\alpha > \gamma$, there exists uncountably many irrationals in $B(j)\backslash K(\gamma)$ which do not belong to the nowhere dense subset $\cup\{K(\beta) : \beta < \alpha\}$ of $\mathbb{R}$.

There must then exist some ordinal $\mu > \gamma$ such that $k \notin \cup\{K(\beta) : \beta < \mu\}$, for some irrational k in $B(j)\backslash K(\gamma)$.

Let δ be the least ordinal $\geq \mu$, such that $K(\delta) = \{k\}$ where $k \notin \cup\{K(\beta) : \beta < \delta\}$.

Then $x_\delta = k \in B(j)$. So T is dense in $\mathbb{J}$. Then T is dense in $\mathbb{R}$, as required.