25 / Perfect functions

Abstract. *In this section we introduce the notion of a "perfect function". After providing a formal definition we derive two of its most well-known characterizations. Our brief discussion of perfect functions will involve notions discussed in our study of singular functions.*

25.1 Introduction.

We will discuss in this chapter a particular type of function referred to as a "perfect function" (not that there is nothing particularly "perfect" about it). It's nomenclature may be due to the fact that it is considered to be a "nice" function. It is continuous (needs not be real-valued), is closed and often behaves like a homeomorphism, except for the fact that it is not necessarily one-to-one, nor need it be open. It is known to exhibit a few interesting properties. For example, it can be shown that a perfect function $f : S \to T$ mapping S onto T, carries over most topological properties which can be described in terms of open subcovers, from S to T. Of particular interest is the fact that a perfect function carries over the completely regular property from it domain to it codomain. Also, its inverse, $f^{\leftarrow}$, carries over the compact property from its codomain to its domain.

It's study involves many of the topological concepts and techniques studied up to now in various chapters of this text. This will help keep these concepts fresh in the readers mind thus underlining their importance. This, by itself, constitutes one valid reason why one might take a bit time and energy to study it.

25.2 Perfect function: Definition.

Recall that, for a function, $f : S \to T$, the *fibres of f* refers to the elements of the set

$$\{\, f^{\leftarrow}(y) : y \in f[S] \,\} \subseteq \mathscr{P}(S)$$

Definition 25.1 Let S and T be completely regular spaces and $f : S \to T$ be a continuous function mapping S *onto* T. We say that f is a *compact* *mapping* if its fibres are all compact (subsets of S). That is, $f^{\leftarrow}(y)$ is compact for each y in the range of f.

We say that the continuous function, $f : S \to T$, is a *perfect function* if f is both a closed function and a compact mapping.[1]

Theorem 25.2 Let $f : S \to T$ be a perfect function mapping the space S onto the space T. Then T is a compact space if and only if S a compact space.

Proof: We are given that $f : S \to T$ is a perfect function mapping S onto T. If S is compact then, since f is perfect it is continuous and so T must be compact.

Conversely, suppose T is compact. Let $\mathscr{U}$ be an open cover of S. To show that S is compact it suffices to show that $\mathscr{U}$ has a finite subcover of S.

Let $p \in T$. Then, since f is perfect, $f^{\leftarrow}(p)$ is compact. Let $\mathscr{U}_p$ be a subset of $\mathscr{U}$ which covers $f^{\leftarrow}(p)$. Then $\mathscr{U}_p$ has a finite subcover, $\{U_{p_i} : i \in F_p\}$, of $f^{\leftarrow}(p)$. Since f is a closed map, for each $p \in T$,

$$K_p = f[S \backslash \cup \{U_{p_i} : i \in F_p\}]$$

is closed in T, such that

$$p \in H_p = T \backslash K_p$$

where, for each $p \in T$, H_p is an open neighborhood of p. Then $\mathscr{H} = \{H_p : p \in T\}$ forms an open cover of the compact space T. Let $\mathscr{H}_M = \{H_q : q \in M \subseteq T\}$ be a finite subcover of T where M is a finite subset of T. For each $q \in M$,

$$\begin{aligned} f^{\leftarrow}(q) \subseteq f^{\leftarrow}[H_q] &= f^{\leftarrow}[T \backslash K_q] \\ &= S \backslash f^{\leftarrow}[K_q] \\ &= S \backslash f^{\leftarrow}[f[S \backslash \cup \{U_{q_i} : i \in F_q\}]] \\ &= S \backslash [S \backslash \cup \{U_{q_i} : i \in F_q\}] \\ &= \cup \{U_{q_i} : i \in F_q\} \end{aligned}$$

So, for each $q \in M \subseteq_{\text{finite}} T$, $\{U_{q_i} : i \in F_q\}$ is a finite subcover of the compact set, $f^{\leftarrow}(q)$. Since $\{H_q : q \in M\}$ covers T then

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[1] Some authors may not require that perfect functions be continuous.

$\{U_{q_i} : i \in F_q\}_{q \in M}$ is a finite subcover of S.

We conclude that, if T is compact then S is compact.

Theorem 25.3 Let $f : S \to T$ be a perfect function mapping the Hausdorff space S onto the space T. Then T is a paracompact space if and only if the space S is paracompact.

Proof: ($\Rightarrow$) Suppose T is paracompact.

Let $\mathscr{U}$ be an open cover of S. We are required to show that $\mathscr{U}$ has an open refinement which covers S.

Step #1. We first construct an open refinement of $\mathscr{U}$ which covers S.

Let $p \in T$. Then, since f is perfect, $f^{\leftarrow}(p)$ is compact. Let $\mathscr{U}_p$ be a subset of $\mathscr{U}$ which covers $f^{\leftarrow}(p)$. Then $\mathscr{U}_p$ has a finite subcover,

$$\mathscr{U}_{p_F} = \{U_{p_i} : i \in F_p\} \subseteq_{\text{finite}} \mathscr{U}$$

of $f^{\leftarrow}(p)$. Since f is a closed map, for each $p \in T$,

$$K_p = f[S \backslash \cup \{U_{p_i} : i \in F_p\}]$$

is closed in T, such that

$$p \in H_p = T \backslash K_p$$

where for each $p \in T$, H_p is an open neighborhood of p in T. Then

$$\mathscr{H} = \{H_p : p \in T\}$$

forms an open cover of the paracompact space T.
Then $\mathscr{H}$ has an open locally finite refinement, say

$$\mathscr{H}_M = \{M_q : q \in T\}$$

that covers T such that $q \in M_q \subseteq H_q$ (that is, M_q refines H_q).

For each $q \in T$,

$$\begin{aligned} f^{\leftarrow}(q) \subseteq f^{\leftarrow}[M_q] &\subseteq f^{\leftarrow}[H_q] \\ &= f^{\leftarrow}[T \backslash K_q] \\ &= S \backslash f^{\leftarrow}[K_q] \\ &= S \backslash f^{\leftarrow}[f[S \backslash \cup \{U_{q_i} : i \in F_q\}]] \\ &= S \backslash [S \backslash \cup \{U_{q_i} : i \in F_q\}] \\ &= \cup \{U_{q_i} : i \in F_q\} \end{aligned}$$

So, for each $q \in T$,

$$f^{\leftarrow}(q) \subseteq f^{\leftarrow}[M_q] \subseteq \cup\{f^{\leftarrow}[M_q] \cap U_{q_i} : i \in F_q\}$$

where $U_{q_i} \in \mathscr{U}$. So

$$\mathscr{W} = \{f^{\leftarrow}[M_q] \cap U_{q_i} : i \in F_q\}_{q\in T}$$

is an open refinement of $\mathscr{U}$ which covers S. This completes Step #1.

Step #2. We now need only show that $\mathscr{W}$ is locally finite.

Let $x \in S$. Then $x \in f^{\leftarrow}(q_x) \subseteq f^{\leftarrow}[M_{q_x}]$ for some $q_x \in T$. Since $\{M_q : q \in T\}$ is locally finite, there exists an open neighborhood D of q_x in T such that D meets at most finitely many M_q's. Since f is continuous, then $f^{\leftarrow}[D]$ is an open neighborhood of $f^{\leftarrow}(q_x)$. Since $f^{\leftarrow}[D \cap M_q] = f^{\leftarrow}[D] \cap f^{\leftarrow}[M_q]$ then $f^{\leftarrow}[D] \cap f^{\leftarrow}[M_q] = \varnothing$ if and only if $D \cap M_q = \varnothing$. So the open neighborhood, $f^{\leftarrow}[D]$, meets at most finitely many elements of $\mathscr{W}$. So $\mathscr{U}$ has an open locally finite refinement, $\mathscr{W}$, which covers S. We conclude that, for the perfect function $f : S \to T$, if T is paracompact then S is paracompact.

($\Leftarrow$) Let $f : S \to T$ be a perfect function mapping S onto T. Suppose S is Hausdorff and paracompact. Then S is regular. We are required to show that T is paracompact.

Let $\mathscr{K} = \{K_p : p \in T\}$ be an open cover of T. We are required to show that $\mathscr{K}$ has an open refinement which covers T.

For each $p \in T$, let $U_p = f^{\leftarrow}[K_p]$. Hence, for each $p \in T$, $f^{\leftarrow}(p) \subseteq U_p$, so $\mathscr{U} = \{U_p : p \in T\}$ is an open covering of S.

For each $p \in T$, let $\mathscr{U}_p$ be a subset of $\mathscr{U}$ which covers $f^{\leftarrow}(p)$. Since f is perfect, $f^{\leftarrow}(p)$ is compact. Then $\mathscr{U}_p$ has a finite subcover,

$$\mathscr{U}_{p_F} = \{U_{p_i} : i \in F_p\} \subseteq_{\text{finite}} \mathscr{U}$$

of $f^{\leftarrow}(p)$. Then $f^{\leftarrow}(p) \subseteq \cup\{U_{p_i} : i \in F_p\}$. Furthermore,

$$S = \cup\{f^{\leftarrow}(p) : p \in T\} \subseteq \cup_{p\in T}[\cup_{i\in F_p}\{U_{p_i}\}]$$

Then $\mathscr{W} = \{U_{p_i} : i \in F_p\}_{p\in T}$ is an open cover of S. See that $\mathscr{W} \subseteq \mathscr{U}$.

Since S is paracompact, $\mathscr{W}$ has an open locally finite refinement,

$$\mathscr{V} = \{V_{p_i} : i \in F_p\}_{p\in T}$$

which covers S, where $V_{p_i} \subseteq U_{p_i}$. For each $p \in T$,

$$f^{\leftarrow}(p) \subseteq \cup\{V_{p_i} : i \in F_p\} \subseteq \cup\{U_{p_i} : i \in F_p\}$$

For each $p \in T$, we can choose

$$t_p \in f^{\leftarrow}(p) \cap V_{p_j}$$

for some $j \in F_p$.

Then $t_p \in V_{p_j} \subseteq U_{p_j} = f^{\leftarrow}[F_{p_j}]$.

Since S is Hausdorff (hence regular), there exist an open subset M_p of S such that

$$t_p \in M_p \ \subseteq\ \mathrm{cl}_S M_p \subseteq V_{p_j} \ \subseteq\ U_{p_j} = f^{\leftarrow}[F_{p_j}]$$

Since $\mathscr{V}$ is locally finite, then the collection $\mathscr{M} = \{M_p : p \in T\}$ is an open locally finite refinement of $\mathscr{W}$ which covers S. Furthermore, the collection

$$\mathscr{M}^* = \{\mathrm{cl}_S M_p : p \in T\}$$

is a closed locally finite refinement of $\mathscr{W}$ which covers S.

Claim: We claim that

$$f[\mathscr{M}^*] = \{f[\mathrm{cl}_S M_p] : p \in T\}$$

is a closed refinement of $\mathscr{K}$ which covers T.

Proof of claim. Suppose $p \in T$.

$$\begin{array}{rcl} p & = & f(t_p) \\ & \in & f[M_p] \\ & \subseteq & f[\mathrm{cl}_S M_p] \quad \text{(An element of } f[\mathscr{M}^*]\text{)} \\ & \subseteq & f[V_{p_j}] \\ & \subseteq & f[U_{p_j}] \\ & = & f[f^{\leftarrow}[K_{p_j}]] \\ & = & K_{p_j} \quad \text{(An element of } \mathscr{K}\text{)} \end{array}$$

So $f[\mathscr{M}^*]$ refines $\mathscr{K}$.

Claim: The collection $f[\mathscr{M}^*] = \{f[\mathrm{cl}_S M_p] : p \in T\}$ is locally finite.

Proof of claim. Let $d \in T$. It suffices to show that there is an open neighborhood Q of p which meets at most finitely many elements of $f[\mathscr{M}^*]$. Since $\mathscr{M}^*$, is locally finite, for each $x \in f^{\leftarrow}(p)$ there exists an open neighborhood D_x of x such that D_x meets at most finitely many elements of $\mathscr{M}^*$. Since $f^{\leftarrow}(p)$ is compact there exists a finite subcover $\{D_{x_i} : i \in F\}$ of $f^{\leftarrow}(d)$.

Then $D = \cup\{D_{x_i} : i \in F\}$ is an open neighborhood of $f^{\leftarrow}(d)$ which meets at most finitely many elements, say $\mathscr{A} = \{\mathrm{cl}_S M_j : j \in F_K\}$, of $\mathscr{M}^*$. If $\mathrm{cl}_S M_p \in \mathscr{M}^* \backslash \mathscr{A}$ then

$$\mathrm{cl}_S M_p \;\subseteq\; S\backslash D \;\subseteq\; S\backslash f^{\leftarrow}(p)$$

Then

$$E = \cup\{\mathrm{cl}_S M_p : \mathrm{cl}_S M_p \in \mathscr{M}^* \backslash \mathscr{A}\} \;\subseteq\; S\backslash D \;\subseteq\; S\backslash f^{\leftarrow}(p)$$

Since $\mathscr{M}$ is locally finite, by Theorem 6.17,

$$E = \cup\{\mathrm{cl}_S M_p : \mathrm{cl}_S M_p \in \mathscr{M}^* \backslash \mathscr{A}\} = \mathrm{cl}_S[\cup\{M_p : \mathrm{cl}_S M_p \in \mathscr{M}^* \backslash \mathscr{A}\}]$$

so the set E is a closed subset of S which misses $f^{\leftarrow}(p)$.

Then $f[E]$ is a closed subset of T which misses $f[f^{\leftarrow}(p)] = p$.

See that $f[E] = \cup\{f[\mathrm{cl}_S M_p] : \mathrm{cl}_S M_p \in \mathscr{M}^* \backslash \mathscr{A}]\}$.

There exists an open neighborhood Q of p such that $Q \cap f[E] = \varnothing$.

Then only the elements in $\{f[\mathrm{cl}_S M_p] : \mathrm{cl}_S M_p \in \mathscr{A}]\}$ can meet Q. So $f[\mathscr{M}^*]$ forms a closed locally finite collection of sets which covers T.

By Theorem 19.5, T is paracompact.

Given a perfect function, $f : S \to T$, if we compactify the range T of f to αT the following characterization of a perfect function will prove to be useful.

Theorem 25.4 Let $f : S \to T$ be a continuous function mapping a non-compact completely regular space, S, *onto* a non-compact completely regular space T. Then the following are equivalent:

(a) The function $f : S \to T$ is perfect.

(b) If αT is any Hausdorff compactification of T and $f^{\beta(\alpha)} : \beta S \rightarrow \alpha T$ is the continuous extension of $f : S \rightarrow \alpha T$, then

$$f^{\beta(\alpha)}[\beta S \backslash S] \subseteq \mathrm{cl}_{\alpha T} f[S] \backslash f[S]$$

(c) If αT is any Hausdorff compactification of T and $f^{\beta(\alpha)} : \beta S \rightarrow \alpha T$ is the continuous extension of $f : S \rightarrow \alpha T$, then

$$f^{\beta(\alpha)}[\beta S \backslash S] \cap f[S] = \varnothing$$

Proof: Let $f : S \rightarrow T$ be a continuous function mapping a non-compact completely regular space, S, *onto* a non-compact completely regular space T.

(a) $\Rightarrow$ (b) We are given that $f : S \rightarrow T$ is perfect, αT is any Hausdorff compactification of T and $f^{\beta(\alpha)} : \beta S \rightarrow \alpha T$ is the continuous extension of f to βS where $f^{\beta(\alpha)}[\mathrm{cl}_{\beta S} S] = \mathrm{cl}_{\alpha T} f[S] \subseteq \alpha T$.

We are required to show that $f^{\beta(\alpha)}[\beta S \backslash S] \subseteq \mathrm{cl}_{\alpha T} f[S] \backslash f[S]$.

Suppose $f^{\beta(\alpha)}[\beta S \backslash S] \not\subseteq \mathrm{cl}_{\alpha T} f[S] \backslash f[S]$. Therefore $f^{\beta(\alpha)}[\beta S \backslash S] \cap f[S] \neq \varnothing$. There must then exist a $u \in \beta S \backslash S$ such that $f^{\beta(\alpha)}(u) \in f[S]$.

Let

$$K = f^{\leftarrow}(f^{\beta(\alpha)}(u))$$

Then, given that $f : S \rightarrow T$ is perfect, K is a compact subset of the completely regular subspace $S \cup \{u\}$ of βS where $f^{\beta(\alpha)}[S \cup \{u\}] = f[S]$.

Then there is an open neighborhood, U of K in $S \cup \{u\}$ such that

$$K = \mathrm{cl}_{S \cup \{u\}} K \subseteq U \subseteq \mathrm{cl}_{S \cup \{u\}} U \subset (S \cup \{u\}) \backslash \{u\} \qquad (*)$$

So $u \notin \mathrm{cl}_{S \cup \{u\}} U$. Then $u \in \mathrm{cl}_{S \cup \{u\}} (S \backslash U)$.[2]

We then have,

$$\begin{array}{rcll} f^{\beta(\alpha)}(u) & \in & f^{\beta(\alpha)}[\mathrm{cl}_{S \cup \{u\}}[S \backslash U]] & \\ & \subseteq & \mathrm{cl}_T f[S \backslash U] & \text{(By continuity and Theorem 6.4)} \\ & = & f[S \backslash U] & \text{(Since } f \text{ is a closed function.)} \end{array}$$

Then, there is a point t in $S \backslash U$ such that $f^{\beta(\alpha)}(u) = f(t)$. Then $t \in f^{\leftarrow}(f^{\beta(\alpha)}(u)) = K$.

So $t \in K \cap S \backslash U$, contradicting $K \subseteq U$ (see ((*)).

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[2] Since $u \in \mathrm{cl}_{S \cup \{u\}} S = \mathrm{cl}_{S \cup \{u\}} (S \backslash U \cup U) = \mathrm{cl}_{S \cup \{u\}} (S \backslash U) \cup \mathrm{cl}_{S \cup \{u\}} U$ and $u \notin \mathrm{cl}_{S \cup \{u\}} U$ then $u \in \mathrm{cl}_{S \cup \{u\}} (S \backslash U)$.

Then $f^{\beta(\alpha)}[\beta S\backslash S] \subseteq \mathrm{cl}_{\alpha T}f[S]\backslash f[S]$ as required.

We are done with (a) $\Rightarrow$ (b).

(b) $\Rightarrow$ (a) We are given that $f : S \to T$ is a continuous function which extends to $f^{\beta(\alpha)} : \beta S \to f^{\beta(\alpha)}[\beta S] = \mathrm{cl}_{\alpha T}f[S] \subseteq \alpha T$ such that

$$f^{\beta(\alpha)}[\beta S\backslash S] \subseteq \mathrm{cl}_{\alpha T}f[S]\backslash f[S] \qquad (\dagger)$$

We are required to show that $f : S \to T$ is both a compact and closed function.

Step #1. *The function $f : S \to T$ is compact*: Let $y \in f[S] \subseteq T$. Then $f^{\beta(\alpha)\leftarrow}(y)$ is closed in βS.

Case 1: $f^{\beta(\alpha)\leftarrow}(y) \subseteq S$. Then it is a closed subset of βS so it is a compact subset of S.

Case 2: $f^{\beta(\alpha)\leftarrow}(y) \cap \beta S\backslash S \notin \varnothing$. Then, by hypothesis,

$$f^{\beta(\alpha)}\left[[\, f^{\beta(\alpha)\leftarrow}(y)] \cap \beta S\backslash S\right] = \{y\} \subseteq \mathrm{cl}_{\alpha T}f[S]\backslash f[S]$$

Since $y \in f[S]$ (by hypothesis), this case cannot occur. So $f^{\beta(\alpha)\leftarrow}(y) \subseteq S$. Then $f^{\leftarrow}(y)$ is compact in S. So $f : S \to T$ is a compact function. This concludes step #1.

Step #2. *The function $f : S \to T$ is a closed function*: Let F be a closed subset of S.
See that $\mathrm{cl}_{\beta S}F$ is compact in βS and so $f^{\beta(\alpha)}[\mathrm{cl}_{\beta S}F] \cap T$ is a closed subset of T.
We claim that $f^{\beta(\alpha)}[\mathrm{cl}_{\beta S}F] \cap T = f[F]$.

$$\begin{aligned}
f^{\beta(\alpha)}[\mathrm{cl}_{\beta S}F] \cap T &= f^{\beta(\alpha)}[\ (\mathrm{cl}_{\beta S}F \cap \beta S\backslash S) \cup\ F\ ] \cap T \\
&= \left[f^{\beta(\alpha)}[\ (\mathrm{cl}_{\beta S}F \cap \beta S\backslash S)\ ] \cup f^{\beta(\alpha)}[F]\ \right] \cap T \\
&= \left[f^{\beta(\alpha)}[\ (\mathrm{cl}_{\beta S}F \cap \beta S\backslash S)\ ] \cap T \right] \cup \left[f^{\beta(\alpha)}[F]\ \cap T \right] \\
&\subseteq [\ [\mathrm{cl}_{\alpha T}f[S]\backslash f[S]\ ] \cap T\ ] \cup \left[f^{\beta(\alpha)}[F]\ \cap T \right] \quad \text{(By (†)).} \\
&= \varnothing \cup [\ f[F]\ \cap T\ ] \quad \text{(By our hypothesis)} \\
&= f[F]
\end{aligned}$$

So $f^{\beta(\alpha)}[\mathrm{cl}_{\beta S}F] \cap T \subseteq f[F]$. Since $f[F] \subseteq f^{\beta(\alpha)}[\mathrm{cl}_{\beta S}F] \cap T$, then

$$f^{\beta(\alpha)}[\mathrm{cl}_{\beta S}F] \cap T = f[F]$$

a closed subset of T, as claimed.

So f is a closed map.

By definition, f is a perfect function. We are done with (b) $\Rightarrow$ (a).

(b) $\Leftrightarrow$ (c) Suppose that for αT, a Hausdorff compactification of T and that for $f^{\beta(\alpha)} : \beta S \to \alpha T$ the continuous extension of $f : S \to \alpha T$, Then $f^{\beta(\alpha)}[\beta S \backslash S] \subseteq \mathrm{cl}_{\alpha T} f[S] \backslash f[S]$ if and only if $f^{\beta(\alpha)}[\beta S \backslash S] \cap f[S] = \varnothing$.

Remark. See that the choice of the compactification, αT, of T is not relevant in the proof. We only need the compactification αT of T to justify the existence of the extension $f^{\beta(\alpha)} : \beta S \to \alpha T$.

25.3 Relating a perfect function, f, to its singular set, $S(f)$.

Recall (from page 532) that, if we are given a continuous function $f : S \to T$ mapping a non-compact space, S, into a *compact* space T (not necessarily a singular map), we obtain a compactification of S,

$$\gamma S = S \cup^* S(f)$$

(not necessarily a singular compactification).[3] We showed that,

$$f^{\beta(T)}[\beta S \backslash S] \quad = \quad S(f) = f^{\gamma}[S(f)] \qquad \text{(In Theorem 22.5.)}$$

So

$$\begin{aligned} f^{\beta(T)}[\beta S] &= f^{\beta(T)}[\beta S \backslash S] \cup f[S] \\ &= S(f) \cup f[S] = \mathrm{cl}_T f[S] \end{aligned}$$

where $f[S]$ and $S(f)$ may be disjoint, but not necessarily.

In the case where f is a singular map (that is, $f[S]$ is dense in $S(f)$), then

$$S(f) \cup f[S] = S(f)$$

So, if f is singular and if $f[S]$ is a compact subset of T,

$$S(f) = f[S]$$

We now show that, given the function $f : S \to T$ and the function $f : S \to \alpha T$ for any compactification αT, f is perfect depending, specifically, on how its range, $f[S]$, interacts with its singular set, $S(f)$, of f.

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[3] Remember that the symbol $\cup^*$ is to be interpreted as "adjoined to" not "union".

Theorem 25.5 *A characterization of perfect functions.* Let S and T be *non-compact* completely regular spaces and $f : S \to T$ be a continuous function mapping S *onto* T. Let αT be any compactification of T so that $f : S \to \alpha T$ maps S into αT. Then $f : S \to T$ is a perfect map if and only if

$$f[S] \cap S(f) = \varnothing$$

Proof: We are given that $f : S \to T$ is a continuous function mapping S *onto* T, where S and T are both non-compact and Hausdorff. Let αT be a compactification of T so that $f[S]$ is a subset of αT.

Then the function $f : S \to \alpha T$ mapping S onto $f[S] \subseteq \alpha T$ extends to $f^{\beta(\alpha)} : \beta S \to \alpha T$ where $f^{\beta(\alpha)}[\beta S \backslash S] = S(f)$ (by Theorem 22.5) and

$$\begin{aligned} f^{\beta(\alpha)}[\beta S] &= f^{\beta(\alpha)}[\beta S \backslash S] \cup f[S] \\ &= S(f) \cup f[S] \qquad \text{(Theorem 22.5.)} \\ &= \mathrm{cl}_{\alpha T} f[S] \end{aligned}$$

($\Rightarrow$) Suppose f is perfect. We are required to show that $f[S] \cap S(f) = \varnothing$.

Since f is perfect, then, by Theorem 25.4, (c),

$$f^{\beta(\alpha)}[\beta S \backslash S] \cap f[S] = \varnothing$$

Hence $S(f) \cap f[S] = \varnothing$.

($\Leftarrow$) Conversely, we are given that $f : S \to T$ is a continuous function mapping S *onto* T, αT is any compactification of T and the function $f : S \to \alpha T$ which induces the function $f^{\beta(\alpha)} : \beta S \to \alpha T$. Suppose $f[S] \cap S(f) = \varnothing$. We are required to show that f is perfect.

By Theorem 25.4, (c) $\Rightarrow$ (a), it suffices to show that $f^{\beta(\alpha)}[\beta S \backslash S] \cap f[S] = \varnothing$.

We are given that $f[S] \cap S(f) = \varnothing$. Since $f^{\beta(\alpha)}[\beta S \backslash S] = S(f)$, then

$$f^{\beta(\alpha)}[\beta S \backslash S] \cap f[S] = \varnothing$$

So f is perfect. We are done.

25.4 Examples.

The continuous function f does not necessarily carry over the regularity property of its domain to its range. The following example shows that perfect functions do.

Example 1. Suppose $f : S \to T$ is a perfect function mapping the regular space S into the space T. Show that $f[S]$ is regular.

Solution: Suppose that F is closed in $f[S]$ and $x \in f[S]\setminus F$.

Since f is perfect $f^{\leftarrow}[F]$ is a closed subset of S disjoint from the compact set, $f^{\leftarrow}(x)$. If $a \in f^{\leftarrow}(x)$, regularity of S guarantees the existence of disjoint open neighborhoods U_a and V_a of a and $f^{\leftarrow}[F]$, respectively. Then $\{U_a : a \in f^{\leftarrow}(x)\}$ forms an open cover of the compact set, $f^{\leftarrow}(x)$. Let $\{U_{a_i} : i = 1, 2, 3, \ldots, n\}$ be a finite subcover of $f^{\leftarrow}(x)$. Then $\cup\{U_{a_i} : i = 1, 2, 3, \ldots, n\}$ and $\cap\{V_{a_i} : i = 1, 2, 3, \ldots, n\}$ form disjoint open neighborhoods of $f^{\leftarrow}(x)$ and $f^{\leftarrow}[F]$, respectively. Then, since f is closed,

$$f[S] \setminus f[S\setminus\cup\{U_{a_i}\}_{i=1,\ldots,n}] = f[S] \setminus f[\cap\{S\setminus U_{a_i}\}_{i=1,\ldots,n}]$$

and

$$f[S] \setminus f[S\setminus\cap\{V_{a_i}\}_{i=1,\ldots,n}] = f[S] \setminus f[\cup\{S\setminus\{V_{a_i}\}_{i=1,\ldots,n}]$$

are disjoint open neighborhoods of x and F, respectively in $f[S]$. So $f[S]$ is regular.

Example 2. Suppose $f : S \to T$ is a perfect function. Show that if F is a closed subset of S then $f|_F$ is perfect on F.

Solution: Since a perfect function, f on S, was defined as being continuous on S, $f|_F$ is continuous on F. Suppose K is closed in F. Then there exists a closed subset K^* in S such that $K^* \cap F = K$. Then, since f is perfect on S, $f[K]$ is closed in $f[S]$. Then $f|_F[K] = f[K^* \cap F] = f[K]$ is closed in both $f[S]$ and $f|_F[F]$. Let $x \in f|_F[F]$. Then $f|_F^{\leftarrow}(x) = f^{\leftarrow}(x) \cap F$, a closed subset of the compact subset $f^{\leftarrow}(x)$. So $f|_F^{\leftarrow}(x)$ is compact. We conclude that $f|_F$ is perfect.

Example 3. Suppose $f : S \to T$ and $g : T \to Z$ are both perfect functions. Show that the composition, $g{\circ}f : S \to Z$, is perfect.

Solution: Since both f and g are continuous closed functions then so is $g{\circ}f$. If $x \in Z$ then $(g{\circ}f)^{\leftarrow}(x) = f^{\leftarrow}(g^{\leftarrow}(x))$. Let $K = g^{\leftarrow}(x)$. We know that K is compact in T. To show that $g{\circ}f$ is perfect it suffices to show that $f^{\leftarrow}[K]$ is compact.

Let $\mathscr{U}$ be an open cover of $f^{\leftarrow}[K]$. For $a \in K$, $f^{\leftarrow}(a)$ is a compact subset of $f^{\leftarrow}[K]$. Let $\mathscr{U}_a = \{U_i : i = 1, 2, \ldots, n\}$ be a finite subset of $\mathscr{U}$ such that $\mathscr{U}_a$ covers $f^{\leftarrow}(a)$. Let $C_a = \cup \mathscr{U}_a \subseteq \mathscr{U}$. Then

$$f^{\leftarrow}[K] \subseteq \cup\{C_a : a \in K\} = \mathscr{U}$$

Then $f[S \setminus C_a]$ is closed in T. Hence $T \setminus f[S \setminus C_a]$ is an open neighborhood of a in T. Let

$$D_a = T \setminus f[S \backslash C_a]$$

Then $\mathscr{D} = \{D_a : a \in K\}$ forms an open cover of compact K in T. That is

$$K \subseteq \cup \mathscr{D}$$

Then $\mathscr{D}$ contains a finite subset $\{D_{a(i)} : i = 1, 2, \ldots, k\}$ such that

$$K \subseteq \cup\{D_{a(i)} : i = 1, 2, \ldots, k\} = \cup\{T \setminus f[S \setminus C_{a(i)}] : i = 1, 2, \ldots, k\}$$

It follows that $f^{\leftarrow}[K] \subseteq \cup\{C_{a(i)} : i = 1, 2, \ldots, k\}$. So $f^{\leftarrow}[K]$ is covered by a finite subset of the cover $\mathscr{U}$.

We conclude that $f^{\leftarrow}[K]$ is compact. So $g{\circ}f$ is perfect.

Example 4. Suppose S is completely regular and $\mathscr{F} = C^*(S)$. We showed in the example found on page 543 that $e_{\mathscr{F}}[S] \cap S(e_{\mathscr{F}}) = \varnothing$. Then $e_{\mathscr{F}}$ is a perfect function.

Example 5. Suppose S is non-compact completely regular and *connected*. Let $f : S \to \mathbb{R}$ be a continuous real-valued bounded function on S. Then $f[S]$ is a bounded interval. Show that f is a perfect function if and only if $f[S]$ is not a closed interval.

Solution: We are given that non-compact S is connected and $f : S \to \mathbb{R}$ belongs to $C^*(S)$.

Suppose f is a perfect function with a non-compact connected domain. We are required to show that $f[S]$ is not a closed interval in $\mathbb{R}$. Suppose $f[S]$ is a closed bounded interval. Say, $f[S] = [a, b]$. Then $f[S]$ is compact. We have shown that if the range of a perfect function is compact then so is the domain. Since the domain is non-compact by hypothesis, then $f[S]$ cannot be a closed interval.

Conversely, suppose that the bounded interval, $f[S]$, is not closed in $\mathbb{R}$. That is, suppose

$$f[S] = [a, b) \text{ or } f[S] = (a, b] \text{ or } f[S] = (a, b)$$

We are required to show that f must then be a perfect function on S. We consider the case where $f[S] = [a, b)$. That is, $b \notin f[S]$. Then

$$f^{\beta}[\beta S] = \mathrm{cl}_{\mathbb{R}} f[S] = [a, b] = f[S] \cup^{*} S(f)$$

Since $b \notin f[S]$, $\{b\} = S(f)$. Then $S(f) \cap f[S] = \varnothing$ and so, by Theorem 25.5, f is perfect.

Similarly, if $f[S] = (a, b)$ then $\{a, b\} = S(f)$ and if $f[S] = (a, b]$ then $\{a\} = S(f)$. So, in each case, $S(f) \cap f[S] = \varnothing$ and so f is perfect.

Example 6. Let $f = \left(\frac{2}{\pi}\right)\arctan$. We then obtain the function $f : \mathbb{R} \to [-1, 1]$. Show that f is perfect.

Solution: We could argue that, since $\mathbb{R}$ is connected and $f[\mathbb{R}] = (-1, 1)$ is not a closed interval then, by the example above, f is perfect.

We could also argue as follows: The function, f, maps the connected interval $(-\infty, \infty)$ one-to-one onto $T = (-1, 1)$. The function, $f : \mathbb{R} \to [-1, 1]$, is real-valued and bounded. We easily verify that $\{-1, 1\} = S(f)$. Then $f[S] \cap S(f) = (-1, 1) \cap \{-1, 1\} = \varnothing$. Then f is perfect.

Example 7. Let $f : \mathbb{N} \to \beta\mathbb{R}$ be the one-to-one continuous function mapping $\mathbb{N}$ onto $\mathbb{Q}$. We can easily see that f is a compact function (since f is one-to-one onto $\mathbb{Q}$). We now show that $f : \mathbb{N} \to f[\mathbb{N}] = \mathbb{Q}$ is not closed. Let $D = \{x_i : i \in \mathbb{N}\}$ be a sequence of distinct rational numbers converging to a unique rational number x. The set $f^{\leftarrow}[D]$ is closed in $\mathbb{N}$ but $f[f^{\leftarrow}[D]] = D$ is not closed in $\mathbb{Q}$ since it does not contain its unique limit point, x. So f is not a closed function. By definition, f is not a perfect function.

We can arrive at the same conclusion by arguing as follows. In Theorem 22.16 we show that the one-to-one continuous function $f : \mathbb{N} \to \beta\mathbb{R}$ mapping $\mathbb{N}$ onto $\mathbb{Q}$ pulls back any open neighborhood of a point in $\mathbb{Q}$ to an unbounded subset of $\mathbb{N}$. Then, for $f : \mathbb{N} \to \beta\mathbb{R}$, $S(f) = \beta\mathbb{R}$. Since $f[\mathbb{N}] \subseteq S(f)$, f induces a singular compactification, hence from Theorem 25.5, f is not a perfect map.

Example 8. Suppose S is a non-pseudocompact non realcompact space. An example of such a space S is provided on page 601 where we exhibit such a space such that $S \subset vS \subset \beta S$. It is also shown in that example that

$$e_{C(S)}[S] \,\cap\, e_{C(\beta S,\, \omega\mathbb{R})}[\beta S \backslash S] = \varnothing$$

Then, by Theorem 25.4, the function $e_{C(S)} : S \to \prod_{f \in C(S)} \mathbb{R}_f$ is a perfect function.

Example 9. Let $\{S_i : i \in I\}$ be an indexed set of topological spaces and $S = \prod_{i \in I} S_i$ be a corresponding product space. Verify that, for any $k \in I$, the projection map $\pi_k : S \to S_k$ is a perfect function if and only if $\prod_{i \in I \backslash \{k\}} S_i$ is a compact set.

Solution: ($\Rightarrow$) Suppose $S = \prod_{i \in I} S_i$ is a product space and, for any $k \in I$, the projection map $\pi_k : S \to S_k$ is a perfect function. We are required to show that $\prod_{i \in I \backslash \{k\}} T_i$ is compact.

If $a \in S_k$ then $\pi_k^{\leftarrow}(a) = \prod_{i \in I} T_i$ where $T_k = \{a\}$ and $T_i = S_i$ for all $i \neq k$. By hypothesis, $\prod_{i \in I} T_i$ is compact. By invoking the statements in the Theorems 7.20 and 7.21, $\prod_{i \in I} T_i$ is homeomorphic to $T_k \times \prod_{i \in I \backslash \{k\}} T_i$. Since T_k is a singleton set, $\prod_{i \in I \backslash \{k\}} T_i$ is homeomorphic to $\prod_{i \in I} T_i$. Since $\prod_{i \in I} T_i$ is compact then $\prod_{i \in I \backslash \{k\}} T_i$ is compact, as required.

($\Leftarrow$) Let $k \in I$ and suppose $\prod_{i \in I \backslash \{k\}} S_i$ is a compact set. Let $\pi_k : S \to S_k$ be the projection map, $\pi_k(< x_i >) = x_k$. We must show that the fibres of π_k is are compact sets and that π_k is a closed map.

If $a \in S_k$, $\pi_k^{\leftarrow}(a) = \{< x_i > \in \prod_{i \in I} S_i : x_k = a\}$ is homeomorphic to $\prod_{i \in I \backslash \{k\}} S_i$ (since $\pi_k[\pi_k^{\leftarrow}(a)]) = \{a\}$ with all other factors equal to S_i). So $\pi_k^{\leftarrow}(a)$ is compact.

Let $\pi_k : S \to S_k$ be the projection map, $\pi_k(< x_i >) = x_k$. We must show that $\pi_k : \prod_{i \in I} S_i \to S_k$ is a closed map. From Theorems 7.20 and 7.21 we can deduce that $S = \prod_{i \in I} S_i$ is homeomorphic to $S_k \times \prod_{i \in I \backslash \{k\}} S_i$. Let $T = \prod_{i \in I \backslash \{k\}} S_i$. In the example on page 337, we show that, given the product $S \times T$, if the space T is compact the projection map, $\pi_S : S \times T \to S$, is a closed map. When applied to this particular situation we obtain π_k is a closed map.

We conclude that $\pi_k : \prod_{i \in I} S_i \to S_k$ is perfect.

We end this chapter with this somewhat surprising result.

Theorem 25.6 Let $f : S \to T$ and $g : T \to Y$ be two continuous functions on completely regular spaces S and T.

1. Suppose $k = g{\circ}f$ is perfect.
   (a) Then both f and g are compact functions.
   (b) The function g is a closed function.
   (c) The function f is a closed function.
2. The function $k = g{\circ}f$ is perfect if and only if both f and g are perfect.

Proof: We are given that $k = g{\circ}f$ is a perfect function where $f : S \to T$ and $g : T \to Y$ are both continuous.

Part 1. Suppose $k = g{\circ}f$ is perfect.

(a) We are required to show that the fibres of f and of g are compact. For $t \in T$, $g(t) \in Y$. Then, since k is perfect, $k^{\leftarrow}(g(t))$, is a compact subset of S. It easily follows that $f^{\leftarrow}(t)$ is a closed subset of the compact set $k^{\leftarrow}(g(t))$. This implies that f is a compact function. We now verify that this is also the case for g. Let $y \in Y$. Then $g^{\leftarrow}(y)$ is closed subset of the compact set $f[k^{\leftarrow}(y)]$. We conclude that g is also a compact function.

(b) Suppose F is a closed subset of T. We are required to show that $g[F]$ is closed in Y. Then $g[F] = k[f^{\leftarrow}[F]]$. Continuity of f guarantees that $f^{\leftarrow}[F]$ is closed in S. Since k is perfect, it is closed, so $k[f^{\leftarrow}[F]]$ is closed in Y. So g is a closed function, as required.

(c) Suppose K is a closed subset of S. We are required to show that $f[K]$ is closed in T.

Suppose not. That is, suppose there exists, $p \in \mathrm{cl}_T f[K]$, but $p \notin f[K]$. We seek a contradiction. Then, since g is continuous and k is closed,

$$g(p) \in g[\mathrm{cl}_T f[K]] \subseteq \mathrm{cl}_Y g[f[K]] = \mathrm{cl}_Y (g{\circ}f)[K] = k[K]$$

Since $g(p) \in k[K]$ then $k^{\leftarrow}(g(p)) \subseteq K$.

$$\begin{aligned} f[k^{\leftarrow}[g(p)] &= f[K \cap k^{\leftarrow}(g(p))] \\ &\subseteq f[K] \cap f[(g{\circ}f)^{\leftarrow}(g(p)] \\ &= f[K] \cap f[f^{\leftarrow}[g^{\leftarrow}(g(p)] \\ &= f[K] \cap g^{\leftarrow}(g(p)) \end{aligned}$$

Since $p \notin f[K]$, $f[K] \cap g^{\leftarrow}(g(p))$ is non-empty compact subset of T disjoint from $\{p\}$. Then there exists disjoint T-open neighborhoods, U and V, such that

$$\begin{array}{rcl} \{p\} & \subseteq & U \\ f[k^{\leftarrow}(g(p))] \subseteq [f[K] \cap g^{\leftarrow}(g(p))] & \subseteq & V \end{array}$$

Claim: We claim that no such pair of disjoint open neighborhoods, U and V, can exist. That is, that $U \cap V \neq \varnothing$.

Proof of claim:
See that

$$[\,K \backslash f^{\leftarrow}[V]\,] \;\cap\; k^{\leftarrow}(g(p)) = \varnothing \quad \text{(Check!)}$$

Now, since k is a closed map, $k[K \backslash f^{\leftarrow}[V]]$ is closed in Y.

Hence $Y \backslash k[K \backslash f^{\leftarrow}[V]]$ is a Y-open neighborhood of $g(p)$.

Given that U is an open neighborhood of $p \in \mathrm{cl}_T f[K] \setminus f[K]$, then $p \in \mathrm{cl}_T(U \cap f[K])$. Hence $g(p) \in g[\mathrm{cl}_T[U \cap f[K]] \subseteq \mathrm{cl}_Y g[U \cap f[K]]$. Then

$$g[U \cap f[K]] \quad \cap \quad Y \backslash k[K \backslash f^{\leftarrow}[V]] \neq \varnothing$$

If $x \in LHS$, $x = g(q)$ for some $q \in U \cap f[K]$ where $g(q) \notin k[K \backslash f^{\leftarrow}[V]]$. Then, for the first case,

$$f^{\leftarrow}(q) \subseteq f^{\leftarrow}[U \cap f[K]] \subseteq f^{\leftarrow}[U] \Rightarrow q \in U$$

For the second case, since $g(q) \notin k[K \backslash f^{\leftarrow}[V]] = g[f[K \backslash f^{\leftarrow}[V]]]$, $q \notin f[K \backslash f^{\leftarrow}[V]]$. Then $f^{\leftarrow}(q) \not\subseteq K \backslash f^{\leftarrow}[V]$. Then $f^{\leftarrow}(q) \subseteq f^{\leftarrow}[V]$. Then $q \in V$. So $q \in U \cap V \neq \varnothing$, as claimed.

This provides us the required contradiction.

The statement in Part 2 follows immediately from (a), (b) and (c) of Part 1 and Example 3 on page 614.

We illustrate another approach to the proof of "$k = g \circ f$ perfect implies both f and g are perfect".

Theorem 25.7 Let $f : S \to T$ and $g : T \to Y$ be two continuous functions on completely regular spaces S and T. If the function $k = g \circ f$ is perfect then both f and g are perfect.

Proof: We are given that $k = g{\circ}f$ is a perfect function where $f : S \to T$ and $g : T \to Y$ are continuous functions on completely regular spaces, S and T, and where both T and Y are compact.

We have already show that, since k is perfect, then $S(k) \cap k[S] = \varnothing$. Then, for any non-empty open subset, U, of $k[S]$, $\mathrm{cl}_S k^{\leftarrow}[U]$ is compact.

Step 1. We first show that $g : f[S] \to g[f[S]]$ is a perfect function. See that,

$$\begin{aligned} k^{\beta(Y)}[\beta S] &= \mathrm{cl}_Y k[S] = \mathrm{cl}_Y g[f[S]] \\ &= k^{\beta(Y)}[\beta S \backslash S] \cup k[S] \\ &= S(k) \cup k[S] = S(k) \cup g[f[S]] \end{aligned}$$

and

$$\begin{aligned} g^{\beta(Y)}[\beta(f[S])] &= \mathrm{cl}_Y g[f[S]] = \mathrm{cl}_Y k[S] = S(k) \cup k[S] \\ &= g^{\beta(Y)}[\beta(f[S]) \backslash f[S]] \cup g[f[S]] \\ &= S(g) \cup g[f[S]] = S(g) \cup k[S] \end{aligned}$$

Since, $g[f[S]] = k[S]$, then $S(k) \cup k[S] = S(g) \cup g[f[S]]$. Then $S(g) = S(k)$. Then $S(g) \cap g[f[S]] = \varnothing$. So the function, $g : f[S] \to g[f[S]] \subseteq Y$, is a perfect function. We are done with Step 1.

Step 2. We now show that f is perfect. Suppose not. That is, suppose there is a point $x \in f^{\beta}[\beta S \backslash S] \cap f[S] = S(f) \cap f[S]$. Then there is $y \in \beta S \backslash S$ and $z \in S$ such that $f^{\beta}(y) = f(z) = x \in f[S]$. Then $g(f^{\beta}(y)) = g(f(z)) = g(x) = k(z) \in k[S]$.

Let U be an open neighborhood of $k(z) = g(f(z))$ in Y. Then, since $k(z) \in U$,

$$z \in k^{\leftarrow}[U] = (g{\circ}f)^{\leftarrow}[U] = f^{\leftarrow}[g^{\leftarrow}[U]]$$

Then $f(z) = x \in g^{\leftarrow}[U]$. Since $f(z) \in S(f)$ then, by definition of $S(f)$, $\mathrm{cl}_S f^{\leftarrow}[g^{\leftarrow}[U]]$ is not compact. Then $\mathrm{cl}_S k^{\leftarrow}[U]$ is not compact, contradicting the fact that k is perfect. The source of the contradiction is our assumption that f is not perfect. So f is perfect and so this completes Step 2 and the theorem.

Concepts review.

1. Define a compact function.

2. Define a perfect function.
3. If $f : S\mathbb{R}T$ is a perfect function and T is compact what can we say about S?
4. Provide two characterizations of a perfect function.
5. Provide a characterization of the realcompact property involving a perfect function.
6. Provide a characterization of a perfect function $f : S \to K$ in terms of a singular set.
7. Is $e_{C^*(S)} : S \to \prod_{f \in C^*(S)} f[S]$ a perfect map? Why?