24 / Realcompact spaces

Abstract. *In this section we introduce the set of all points in βS called the "real points" of a space S. The set of all real points of S is denoted by υS. Given a space S, the extension of S, υS, is referred to as the Hewitt-Nachbin realcompactification of S. If $\upsilon S = S$, then S is said to be realcompact. We provide a characterization as well as a few examples of realcompact spaces.*

24.1 Realcompact space: Definitions and characterizations.

Suppose S is a completely regular topological space and $f : S \to \mathbb{R}$ is a function in the set, $C(S)$, of all real-valued continuous functions (including unbounded ones) on S. Let $i : \mathbb{R} \to \omega\mathbb{R}$ be the inclusion function (identity map) which embeds $\mathbb{R}$ into the one-point compactification,

$$\omega\mathbb{R} = \mathbb{R} \cup \{\infty\}$$

defined as, $i(x) = x$ (where ∞ represents a point not in $\mathbb{R}$). We can combine the two functions f and i to define the function $f_+ = i \circ f$, where $f_+ : S \to \omega\mathbb{R}$ maps S into the compact set $\omega\mathbb{R}$. In this case, we have practically identical functions, f_+ and f, except that the codomain of f_+ is $\omega\mathbb{R}$. We have shown in Theorem 21.5 that any continuous function, $g : S \to K$, mapping a completely regular space, S, into a compact space K extends to a function, $g^{\beta(K)} : \beta S \to K$. In this context, we will denote the extension of $f_+ : S \to \omega\mathbb{R}$ to βS as

$$f^{\beta(\omega)} : \beta S \to \omega\mathbb{R}$$

That is, if $f \in C(S)$ (possibly unbounded), f extends to $f^{\beta(\omega)} \in C(\beta S, \omega\mathbb{R})$.

In the case where f is bounded, the extension $f^{\beta(\omega)} : \beta S \to \omega\mathbb{R}$ maps the compact space βS onto the compact space

$$f^{\beta(\omega)}[\beta S] = f^{\beta(\omega)}[\mathrm{cl}_{\beta S} S] = \mathrm{cl}_{\omega\mathbb{R}} f[S] = \mathrm{cl}_{\mathbb{R}} f[S]$$

a compact set in $\mathbb{R} \subset \omega\mathbb{R}$. So, in the case of a bounded function f, $f^{\beta(\omega)} : \beta S \to \omega\mathbb{R}$ and the usual, $f^{\beta} : \beta S \to \mathbb{R}$, both represent the same function.

Of course, if $f : S \to \mathbb{R}$ is unbounded,

$$\begin{aligned} f^{\beta(\omega)}[\beta S] &= f^{\beta(\omega)}[\beta S \backslash S \;\; \cup \;\; S] \\ &= f^{\beta(\omega)}[\beta S \backslash S] \cup f[S] \\ &\subseteq f^{\beta(\omega)}[\beta S \backslash S] \; \cup \; \mathbb{R} \end{aligned}$$

where $f[S]$ is non-compact in $\mathbb{R}$ (since $f[S]$ is unbounded). So $f^{\beta(\omega)}$ must map at least one point in $\beta S \backslash S$ to ∞. We stress *at least* one point, since $f^{\beta(\omega)\leftarrow}(\infty)$ need not be a singleton set in βS. We then have the disjoint union,

$$\beta S = f^{\beta(\omega)\leftarrow}[\mathbb{R}] \; \cup \; f^{\beta(\omega)\leftarrow}(\infty)$$

So, if $C(S)$ has some unbounded functions,

$$C(\beta S, \omega\mathbb{R}) \nsubseteq C(\beta S)$$

Real points in βS.

Given $f \in C(S)$, we will want to distinguish those points in βS which belong to $f^{\beta(\omega)\leftarrow}(\infty)$ from those which belong to $f^{\beta(\omega)\leftarrow}[\mathbb{R}]$. For a given $f \in C(S)$, we then define $\upsilon_f S$ [1] as,

$$\upsilon_f S = \beta S \backslash f^{\beta(\omega)\leftarrow}(\infty) = f^{\beta(\omega)\leftarrow}[\mathbb{R}]$$

Clearly, $S \subseteq \upsilon_f S \subseteq \beta S$, for all f.

For a bounded function $f \in C^*(S)$,

$$\begin{aligned} \upsilon_f S &= f^{\beta(\omega)\leftarrow}[\mathbb{R}] \\ &= f^{\beta(\omega)\leftarrow}[\mathrm{cl}_{\mathbb{R}} f[S]] \\ &= f^{\beta\leftarrow}[\mathrm{cl}_{\mathbb{R}} f[S]] \\ &= \beta S \end{aligned}$$

So, in the particular case, $f \in C^*(S)$

$$\upsilon_f S = \beta S$$

The function, $f : S \to \mathbb{R}$, in $C(S)$ is one which continuously extends to a real-valued function, $f^{\beta(\omega)} : \upsilon_f S \to \mathbb{R}$ in $C(\upsilon_f S)$. The points in $\upsilon_f S$ are commonly referred to as being the set of all *real points of f* (in the sense that $f^{\beta(\omega)}(x)$ has a real number value at each of point

[1] The Greek letter, υ, is pronounced *upsilon*.

x in $\upsilon_f S$). To consider the "real points" associated to all functions, $f \in C(S)$, we define.

$$\upsilon S \;=\; \cap\{\upsilon_f S : f \in C(S)\} \;=\; \cap\{f^{\beta(\omega)\leftarrow}[\mathbb{R}] : f \in C(S)\}$$

Note that every function, $f \in C(S)$, extends continuously to a real-valued function,

$$f^{\beta(\omega)}|_{\upsilon S}$$

on υS. More specifically, each function in $C(S)$ is associated to a unique function in $C(\upsilon S)$.

With these facts in mind, we can now formally define the following related concepts.

Definition 24.1 Let S be a completely regular space.

(a) If $f \in C(S)$, the function $f^{\beta(\omega)} : \beta S \to \omega\mathbb{R}$ represents the extension of f to βS with range $\omega\mathbb{R}$. We define the subset, $\upsilon_f S$, of βS as

$$\upsilon_f S = f^{\beta(\omega)\leftarrow}[\mathbb{R}]$$

If $p \in \beta S \backslash \upsilon_f S$, then $f^{\beta(\omega)}(p) = \infty$. The points in $\upsilon_f S$ are referred to as *real points associated to the function, f.*

(b) We define, the subspace, υS of βS, as

$$\upsilon S = \cap\{\upsilon_f S : f \in C(S)\} = \cap\{f^{\beta(\omega)\leftarrow}[\mathbb{R}] : f \in C(S)\}$$

The elements of $\upsilon S \subseteq \beta S$ are called *real points of S*. Every element of S is a real point. If $p \in \beta S \backslash \upsilon S$, p is not *a real point of S* and so there must be at least one function $f \in C(S)$ such that $f^{\beta(\omega)}(p) = \infty$.

(c) If the space S is such that $S = \upsilon S$, then we refer to S as being a *realcompact space*. A space S is realcompact if and only if it contains all real points. If S is realcompact $\beta S \backslash S$ contains no real points.[2]

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[2]Note that realcompactness of a space, S, is defined only with the assumption that S is completely regular. If $S = \upsilon S$, the space S need not be open in βS and so need not be locally compact.

Based on the above definition, for any completely regular space S,

$$S \subseteq \upsilon S \subseteq \beta S$$

Also, if $x \in \upsilon_f S$ then $f^{\beta(\omega)}(x) \in \mathbb{R}$.

Furthermore, if f is any function in $C(S)$, f can be viewed as belonging to $C(S, \omega\mathbb{R})$ and so extends to a continuous function $f^{\beta(\omega)}|_{\upsilon S} : \upsilon S \to \omega\mathbb{R}$. In fact,

$$f \in C(S) \quad \Rightarrow \quad f^{\beta(\omega)}|_{\upsilon S} \in C(\upsilon S)$$

We will more succinctly denote the function $f^{\beta(\omega)}|_{\upsilon S}$ by

$$f^{\upsilon} = f^{\beta(\omega)}|_{\upsilon S}$$

If we define $C_{\upsilon}(S) = \{f|_S \in C(S) : f \in C(\upsilon S)\}$, then

$$C^*(S) \subseteq C(S) = C_{\upsilon}(S)$$

At one end of the spectrum we have, for example, the class of countably compact spaces (since these spaces have been shown to have only bounded functions). So, for a countably compact space, S, for every function f in $C(S)$, $f^{\beta(\omega)}$ is real-valued on βS; therefore, for any f, every point in $\beta S \backslash S$ is a real point of f. In this case $\upsilon S = \beta S$.

In fact, if S is any pseudocompact space, then $\upsilon S = \beta S$.

At the other end of the spectrum we have the class of all realcompact spaces S where $S = \upsilon S$. In this case, for every point $p \in \beta S \backslash S$, there is some function $f \in C(S)$ such that $f^{\beta(\omega)}(p) = \infty$. That is, only S contains real points. We will eventually show that $\mathbb{R}$ (equipped with the usual topology) is such a space.

Observe that,

$$\begin{aligned}
x \in \upsilon S \quad &\Leftrightarrow \quad f^{\beta(\omega)}(x) \in \mathbb{R} \text{ for all } f \in C(S, \omega\mathbb{R}) \\
&\Leftrightarrow \quad < f^{\beta(\omega)}(x) >_{f \in C(S,\omega\mathbb{R})} \;\in\; \textstyle\prod_{f \in C(S,\omega\mathbb{R})} \mathbb{R}_f \\
&\Leftrightarrow \quad e^{\beta(\omega)}_{C(S,\omega\mathbb{R})}(x) \;\in\; \textstyle\prod_{f \in C(S,\omega\mathbb{R})} \mathbb{R}_f \\
&\Leftrightarrow \quad e^{\beta(\omega)}_{C(S,\omega\mathbb{R})}(x) \;=\; e^{\upsilon}_{C(S)}(x)
\end{aligned}$$

So we can also describe the subset, υS, of βS as follows:

$$\begin{aligned}
\upsilon S \quad &= \quad \{x \in \beta S : f^{\beta(\omega)}(x) \in \mathbb{R} \;\forall\; f \in C(S)\} \\
&= \quad \{x \in \beta S : < f^{\beta(\omega)}(x) >_{f \in C(S)} \;\in\; \textstyle\prod_{f \in C(S)} \mathbb{R}_f\} \\
&= \quad \{x \in \beta S : \; e^{\beta(\omega)}_{C(S)}(x) \in \textstyle\prod_{f \in C(S)} \mathbb{R}_f\} \\
&= \quad e^{\beta(\omega)\leftarrow}_{C(S)} \left[\textstyle\prod_{f \in C(S)} \mathbb{R}_f\right]
\end{aligned}$$

We will require the following concept for our first characterization of a "realcompact space". It refers to a slightly stronger condition on z-ultrafilters. Recall that a z-filter, $\mathscr{F}$ is, by definition, a family of subsets of $Z[S]$ in which every finite subset of $\mathscr{F}$ has non-empty intersection. Countably infinite subfamilies of a z-filter, $\mathscr{F}$ need not have non-empty intersections. So a z-ultrafilter $\mathscr{F}$ in which every countable (possibly infinite) subfamily of $\mathscr{F}$ has non-empty intersection is a "particular" type of z-ultrafilter, called *real*.

Definition 24.2 Let S be a completely regular space. A z-ultrafilter, $\mathscr{F}$, in $Z[S]$ is called a

real z-ultrafilter

if and only if any countable intersection of elements of $\mathscr{F}$ has non-empty intersection.

Those z-ultrafilters $\mathscr{F}$ in $Z[S]$ which contain a countably infinite subfamily of zero-sets which have empty intersection are not "real".

For the following theorem, recall that a space S is compact if and only if every z-ultrafilter is fixed (that is, has non-empty intersection). It shows that a space S is realcompact if and only if its *real* z-ultrafilters are fixed. That is, S is realcompact if and only if any z-ultrafilter in $Z[S]$, in which every countable (possibly infinite) subfamily has non-empty intersection, is a fixed z-ultrafilter.

Theorem 24.3 Let S be a completely regular space.

(a) If $p \in \beta S \backslash \upsilon S$, then $p \in Z(h^{\beta}) \subseteq \beta S \backslash \upsilon S$ for some $h \in C^*(S)$.

(b) If $p \in \beta S \backslash \upsilon S$ then no real z-ultrafilter in $Z[S]$ can converge to p.

(c) The following three statements are equivalent.

   (i) The space S is realcompact. That is, $S = \upsilon S$.

   (ii) For any $p \in \beta S \setminus S$, there is a function, $h \in C^*(S)$, such that $p \in Z(h^{\beta}) \subseteq \beta S \backslash S$.

   (iii) Every *real z-ultrafilter* in $Z[S]$ converges to a point $p \in S$.

Proof: We are given that S is completely regular.

(a) Let $p \in \beta S \backslash \upsilon S$.

We are required to show that $p \in Z(h^{\beta}) \subseteq \beta S \backslash \upsilon S$ for some $h \in C^*(S)$.

Since $p \in \beta S \backslash \upsilon S$, then, by definition of υS, p is *not* a real point of S, in the sense that, for some function, say $g \in C(S)$, $g^{\beta(\omega)}(p) = \infty$. Then g^{υ} is unbounded on υS. For, if g^{υ} was bounded on υS,

$$g^{\beta(\omega)}(p) \in g^{\beta(\omega)}[\mathrm{cl}_{\beta S} S] = g^{\beta(\omega)}[\mathrm{cl}_{\beta S} \upsilon S] = \mathrm{cl}_{\omega \mathbb{R}} g^{\upsilon}[\upsilon S] = \mathrm{cl}_{\mathbb{R}} g^{\upsilon}[\upsilon S] \subseteq \mathbb{R}$$

contradicting $g^{\beta(\omega)}(p) = \infty$.

Let $f^{\upsilon} = (g^{\upsilon})^2 \vee 1$. Then $f^{\upsilon} : \upsilon S \to [1, \infty)$ is continuous, unbounded and *nowhere zero* on υS. Furthermore, f^{υ} extends to $f^{\beta(\omega)} : \beta S \to \omega\mathbb{R}$.

Define the function $h^{\upsilon} : \upsilon S \to \mathbb{R}$ as,

$$h^{\upsilon}(x) = (1/f^{\upsilon})(x) = 1/f^{\upsilon}(x)$$

Then $h^{\upsilon} \in C^*(\upsilon S)$ is bounded by 0 and 1.

Now $h^{\upsilon} : \upsilon S \to \mathbb{R}$ extends continuously to $h^{\beta} : \beta S \to \mathbb{R}$ where

$$h^{\beta}[\beta S] = h^{\beta}[\mathrm{cl}_{\beta S} \upsilon S] = \mathrm{cl}_{\mathbb{R}} h^{\upsilon}[\upsilon S] \subseteq \mathrm{cl}_{\mathbb{R}}(0, 1] = [0, 1]$$

Since $p \in \beta S \backslash \upsilon S$ there is sequence $\{x_i : i \in N\}$ in υS converging to p. Then

$$\{h^{\beta}(x_i) : i \in N\} \quad \longrightarrow \quad h^{\beta}\left(\lim_{x_i \to p} x_i\right) = h^{\beta}(p) \notin (0, 1]$$

So $h^{\beta}(p) = 0$. That is, $p \in Z(h^{\beta})$. Since h^{β} is not zero on υS, then

$$p \in Z(h^{\beta}) \subseteq \beta S \backslash \upsilon S$$

As required.

(b) Suppose p is a point in $\beta S \backslash \upsilon S$. To every point p in βS there corresponds a unique z-ultrafilter, $\mathscr{F}_p$, which converges to p. We are required to show that $\mathscr{F}_p$ is not a real z-ultrafilter. To do this, we are required to show that $\mathscr{F}_p$ contains a countable subfamily of zero-sets with empty intersection in S.

By part (a), $p \in Z(h^{\beta}) \subseteq \beta S \backslash \upsilon S$, for some $h \in C^*(S)$.

See that,

$$Z(h^{\beta}) = \cap\{h^{\beta\leftarrow}[-1/n, 1/n] : n \in \mathbb{N}\backslash\{0\}\}$$

where each $h^{\beta\leftarrow}[-1/n, 1/n]$ is a zero-set in $Z[\beta S]$.[3]

Then $Z(h^\beta)$ is a countable intersection of zero-sets which belong to $\mathscr{F}_p$. Therefore, $\{h^{\leftarrow}[-1/n, 1/n] : n \in \mathbb{N}\backslash\{0\}\}$, is a countable family of zero-sets in the z-ultrafilter, $\mathscr{F}_p$, which converges to p.

Since
$$\cap\{h^{\leftarrow}[-1/n, 1/n] : n \in \mathbb{N}\backslash\{0\}\} = \varnothing$$
$\mathscr{F}_p$ cannot be a real z-ultrafilter, as required.

(c) Let S be a completely regular space.

(i) $\Rightarrow$ (ii) We are given that S is realcompact. Then $\upsilon S = S$.

Let $p \in \beta S\backslash \upsilon S = \beta S\backslash S$.
By part (a) $p \in Z(h^\beta) \subseteq \beta S\backslash S$ for some $h \in C^*(S)$.

(ii) $\Rightarrow$ (i) We are given that every point in $\beta S\backslash S$ belongs to a zero-set entirely contained in $\beta S\backslash S$. We are required to show that S is realcompact. Let $p \in \beta S\backslash S$. It suffices to show that p is not a real point.

To say that p is not a real point means that there is a function $h \in C(S)$ such that $h^{\beta(\omega)}(p) = \infty$. By hypothesis, $p \in Z(f^\beta) \subseteq \beta S\backslash S$ for some $f \in C(S)$. Then f is never 0 on S.

We can then define $h : S \to \omega\mathbb{R}$ as $h = 1/f$. Then h extends to $h^{\beta(\omega)} : \beta S \to \omega\mathbb{R}$. There exists a sequence, $\{x_i : i \in N\}$, in S which converges to $p \in Z(f^\beta)$. Then $\{f(x_i) : i \in N\}$ approaches zero as x_i approaches p. By continuity of $h^{\beta(\omega)}$,

$$\begin{aligned} h^{\beta(\omega)}(p) &= h^{\beta(\omega)}\left(\lim_{x_i \to p}\{x_i\}\right) \\ &= \lim_{x_i \to p}\{h(x_i)\} \\ &= \lim_{x_i \to p}\{1/f(x_i)\} \\ &= \infty \in \omega\mathbb{R} \end{aligned}$$

So p is not a real point. Then, if every point in $\beta S\backslash S$ belongs to a zero-set entirely contained in $\beta S\backslash S$ the subset, $\beta S\backslash S$, cannot contain any real points. So $\upsilon S = S$. By definition, S is realcompact. This proves (ii) $\Rightarrow$ (i).

[3]See that $h^{\beta\leftarrow}[\,[a,b]\,] = \cap\{h^{\beta\leftarrow}[\,(a-1/n, b+1/n)\,]\}$ a countable intersection of open subsets of βS. Then $\cap\{h^{\beta\leftarrow}[\,(a-1/n, b+1/n)\,]\}$ is a G_δ equal to the closed subset $h^{\beta\leftarrow}[\,[a,b]\,]$ in the normal space, βS. In a normal space, closed G_δ's are zero-sets (by Theorem 10.10). So, for each n, $h^{\beta\leftarrow}[-1/n, 1/n]$ is a zero-set in βS.

(ii) $\Rightarrow$ (iii) We are given that every point in $\beta S \backslash S$ belongs to a zero-set entirely contained in $\beta S \backslash S$. We are required to show the limit of any real z-ultrafilter belongs to S.

Since (ii) $\Rightarrow$ (i) our hypothesis implies that no point in $\beta S \backslash S$ is real. So $\upsilon S = S$. By part (b) no real z-ultrafilter converges to a point in $\beta S \backslash \upsilon S = \beta S \backslash S$. Then a real z-ultrafilter can only converge to point in S, as required.

(iii) $\Rightarrow$ (ii) We are given that every *real* z-ultrafilter in $Z[S]$ converges to a point in S. Let $p \in \beta S \backslash S$. We are required to produce a zero-set $Z(h) \in Z[S]$ such that $p \in Z(h^\beta) \subseteq \beta S \backslash S$.

Suppose $\mathscr{Z}_p^\beta = \{Z(f^\beta) : f \in C^*(S), p \in Z(f^\beta)\}$ is the unique z-ultrafilter in $Z[\beta S]$ which converges to p. By hypothesis, the corresponding z-ultrafilter, $\mathscr{Z} = \{Z(f) : f \in C^*(S), p \in Z(f^\beta)\}$, in $Z[S]$ is not real. Then, $\mathscr{Z}$ contains a countable subfamily $\{Z(f_n) : f_n \in D, p \in Z(f_n^\beta\}$ such that

$$\cap\{Z(f_n) : f_n \in D, p \in Z(f_n^\beta\} = \varnothing$$

Let

$$W^\beta = \cap\{Z(f_n^\beta) : f_n \in D, p \in Z(f_n^\beta)\}$$

Then W^β is a zero-set, say $W^\beta(h^\beta)$, in $Z[\beta S]$ such that $p \in W^\beta(h^\beta) \subseteq \beta S \backslash S$, as required.

We add a few remarks in reference to non-compact spaces S.

1) Compare statement (b)

> "*. . . $S = \upsilon S$ if and only if, for any $p \in \beta S \backslash S = \beta S \backslash \upsilon S$, there is a function, $h \in C^*(S)$, such that $p \in Z(h^\beta) \subseteq \beta S \backslash S$.*"

above to the theorem statement in 21.17 which states:

> "*..... S is pseudocompact if and only if no zero-set Z in $Z[\beta S]$ is entirely contained in $\beta S \backslash S$.*"

When viewed together, see what this implies.[4]

[4] If S is pseudocompact, then every function in $C(S)$ is bounded and so $\beta S \backslash \upsilon S$ is empty. Being empty it cannot contain a zero-set of $Z[\beta S]$. On the other hand, if a non-compact space, S, is realcompact, then $\beta S \backslash S = \beta S \backslash \upsilon S$ is non-empty. So $C(S)$ contains unbounded functions. So a non-compact realcompact space cannot be pseudocompact. In this case, $\beta S \backslash S$ contains a zero-set.

2) The statement above also says that $S = \upsilon S$ if and only if every *real* z-ultrafilter in $Z[S]$ is fixed (that is, it converges in $S = \upsilon S$). By definition of *real* z-ultrafilter, this means that a space S is realcompact if and only if every z-ultrafilter in $Z[S]$ which satisfies the countable intersection property (CIP) (meaning countable subfamilies if the z-ultrafilter have non-empty intersection) converges.

In Theorem 24.14, we will be present another important characterization of realcompact spaces.

Example 1. Show that $\mathbb{R}$ equipped with the usual topology is a realcompact space.

Solution: Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined as,

$$f(x) = 1/(1 + |x|)$$

The function f is continuous, bounded and nowhere zero on $\mathbb{R}$, with range, $(0, 1]$. Then $f : \mathbb{R} \to (0, 1]$ continuously extends to $f^\beta : \beta\mathbb{R} \to [0, 1]$.

See that

$$\begin{aligned} f^\beta[\beta\mathbb{R}] &= f^\beta[\mathrm{cl}_{\beta\mathbb{R}}\mathbb{R}] \\ &= \mathrm{cl}_{[0,1]} f[\mathbb{R}] \\ &= \mathrm{cl}_{[0,1]}(0, 1] \\ &= [0, 1] \end{aligned}$$

and

$$\begin{aligned} [0, 1] &= f^\beta[\beta\mathbb{R}] \\ &= f^\beta[\beta\mathbb{R}\backslash\mathbb{R}] \cup f^\beta[\mathbb{R}] \\ &= f^\beta[\beta\mathbb{R}\backslash\mathbb{R}] \cup (0, 1] \end{aligned}$$

So $f^\beta[\beta\mathbb{R}\backslash\mathbb{R}] = \{0\}$. Hence, for the function $f(x) = 1/(1+|x|) \in C^*(\mathbb{R})$, $\beta\mathbb{R}\backslash\mathbb{R} = Z(f^\beta)$.

Then, for any $p \in \beta\mathbb{R}\backslash\mathbb{R}$, there is a function $f \in C^*(\mathbb{R})$ such that $p \in Z(f^\beta) \subseteq \beta\mathbb{R}\backslash\mathbb{R}$.

By Theorem 24.3, c) (ii) $\Leftrightarrow$ (i), $\mathbb{R}$ is realcompact.

24.2 Extending functions to υS.

Let S and T both be a completely regular space. Then S and T can be densely embedded into βS and βT, respectively.

Suppose $f : S \to T$ is a continuous function which maps S onto T. By Theorem 21.5, the function $f : S \to \beta T$ extends to $f^{\beta(\beta T)} : \beta S \to \beta T$. We know that υS is a subspace of βS, hence the restriction, $f^{\beta(\beta T)}|_{\upsilon S} : \upsilon S \to \beta T$, maps υS into βT.

In the following theorem we will show a little bit more. We show that $f^{\upsilon}[\upsilon S] \subseteq \upsilon T$. That is, if $f : S \to T$ is any continuous function mapping S to T, (both completely regular) f will extend to a continuous function f^{υ} which maps υS *into* υT.[6]

Theorem 24.4 If S and T are completely regular spaces and $f : S \to T$ is a continuous function, then f extends continuously to a unique continuous function

$$f^{\upsilon} : \upsilon S \to \upsilon T$$

Proof: We are given that S and T are completely regular and $f : S \to T$ is a continuous function. Since f extends to $f^{\beta(\beta T)} : \beta S \to \beta T$ and $S \subseteq \upsilon S \subseteq \beta S$, then f extends to $f^{\upsilon} : \upsilon S \to \beta T$ (where $f^{\upsilon} = f^{\beta(\beta T)}|_{\upsilon S}$). We are required to show that $f^{\upsilon}[\upsilon S] \subseteq \upsilon T$.

Suppose $h \in C(T)$. Then, if $\omega\mathbb{R} = \mathbb{R} \cup \{\infty\}$,

$$h : T \to \mathbb{R} \text{ will continuously extend to } h^{\omega} : \beta T \to \omega\mathbb{R}$$

Furthermore,

$$(h \circ f) : S \to \mathbb{R} \text{ will continuously extend to } (h \circ f)^{\omega} : \beta S \to \omega\mathbb{R}$$

For $x \in S$, $(h \circ f)^{\omega}|_S(x) = h(f(x)) = (h^{\omega} \circ f^{\beta})|_S(x)$. Since S is dense in βS, then

$$\begin{aligned} (h \circ f)^{\omega} : \beta S &\to \omega\mathbb{R} \\ (h^{\omega} \circ f^{\beta}) : \beta S &\to \omega\mathbb{R} \end{aligned}$$

are equal functions on βS.

Let $r \in \upsilon S$. By definition, r is a real point of the function $h \circ f : S \to \omega\mathbb{R}$. This means that $(h \circ f)^{\omega}(r) = h^{\omega}[f^{\beta}(r)] \neq \infty$. Then $f^{\beta}(r)$ is a real point of h. That is,

$$f^{\upsilon}(r) \in \upsilon_h T$$

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[6] Where $f^{\upsilon} = f^{\beta(\beta T)}|_{\upsilon S}$.

But h was an arbitrarily chosen function in $C(T)$. The point r was also arbitrarily chosen in υS. We can then conclude that

$$f^{\upsilon}[\upsilon S] \subseteq \cap\{\upsilon_h T : h \in C(T)\} = \upsilon T$$

So $f^{\upsilon}[\upsilon S] \subseteq \upsilon T$. This is what we were required to prove.

Corollary 24.5 Suppose S and T are completely regular spaces. If T is realcompact and $f : S \to T$ is a continuous function, then f extends continuously to a unique continuous function $f^{\upsilon} : \upsilon S \to T$. In particular, if $f \in C(S)$, f extends continuously to $f^{\upsilon} \in C(\upsilon S)$.

Proof: This is a special case of the theorem where $T = \upsilon T$. The second part follows from the fact that $\mathbb{R}$ is realcompact as shown in the above example.

From the above corollary, we see that, if S is completely regular, every continuous function $f : S \to \mathbb{R}$ in $C(S)$ extends to $f^{\upsilon} : \upsilon S \to \mathbb{R}$. We know that S is C^*-embedded in βS, but now we can confidently say that

"*S is C-embedded in υS*"

The function

$$\phi : C(S) \to C(\upsilon S)$$

defined as $\phi(f) = f^{\upsilon}$ is then an isomorphism from the ring $C(S)$ onto $C(\upsilon S)$.

24.3 The *Hewitt-Nachbin realcompactification* of a space S.

We have seen that $S \subseteq \upsilon S \subseteq \beta S$. We have also seen that, for every function $f \in C(S)$, the function, $f^{\beta(\omega)}|_{\upsilon S}$, continuously maps υS into $\mathbb{R}$. That is, every function $f \in C(S)$ extends to a function, f^{υ}, in $C(\upsilon S)$.

In what follows we verify how $\beta(\upsilon S)$ compares with βS. Since $\upsilon S \subseteq \beta S$, then $\mathrm{cl}_{\beta S}\upsilon S \subseteq \beta S$. Given that $\mathrm{cl}_{\beta S} S \subseteq \mathrm{cl}_{\beta S}\upsilon S$, then

$$\mathrm{cl}_{\beta S}\upsilon S = \beta S$$

If $f \in C^*(\upsilon S)$ then $f|_S \in C^*(S)$. Since $f|_S$ extends to $(f|_S)^\beta \in C(\beta S)$, so

$$f = [(f|_S)^\beta]|_{\upsilon S}$$

Then $(f|_S)^\beta$ is a continuous extension of f to βS. So υS is C^*-embedded in βS. By Theorem 21.22, $\mathrm{cl}_{\beta S}\upsilon S = \beta(\upsilon S)$. Combining, $\mathrm{cl}_{\beta S}\upsilon S = \beta S$ with $\mathrm{cl}_{\beta S}\upsilon S = \beta(\upsilon S)$ we obtain

$$\beta(\upsilon S) = \beta S$$

One might expect that υS viewed as a subspace of βS is a realcompact space. The following theorem confirms that this is indeed the case.

Theorem 24.6 Suppose S is completely regular. Then υS is a realcompact subspace of βS.

Proof: To show that υS is realcompact, it suffices to show that $\upsilon(\upsilon S) = \upsilon S$. To do this, it suffices to show that the set of all real points of υS is υS.

First note that $\upsilon S \subseteq \upsilon(\upsilon S)$. Then, it suffices to show that $\upsilon(\upsilon S) \subseteq \upsilon S$. Suppose $p \in \upsilon(\upsilon S)$. Then p is a real point of υS. Let $f \in C(S)$. Then there is $g \in C(\upsilon S)$ such that $g = f^\upsilon$. Since p is a real point of υS, then $g^{\beta(\omega)}(p) \in \mathbb{R}$. Since g and f^υ agree on υS, $g^{\beta(\omega)} = f^{\beta(\omega)}$ on βS. So $f^{\beta(\omega)}(p) \in \mathbb{R}$. So p is a real point of S. That is, $p \in \upsilon S$. We have shown that $\upsilon(\upsilon S) \subseteq \upsilon S$.

So

$$\upsilon(\upsilon S) = \upsilon S$$

Then we can view υS as a *realcompact space which densely contains the completely regular space* S.

Given a space S, the topological space, υS, is normally referred to as the

Hewitt-Nachbin realcompactification of S.[7]

[7] Named after the American mathematician, Edwin Hewitt (1920-1999) and the Jewish-Brazilian mathematician Leopoldo Nachbin (1922-1993).

So, instead of referring to υS as being "the set of all real points of S", we will now speak of υS as being "the *realcompactification of* S". If a space, S, is realcompact, then it is its own realcompactification.

If S is pseudocompact, then $C(S) = C^*(S)$ and so every point of βS is a real point of S. Hence, if S is pseudocompact, βS is both the Stone-Čech compactification and the Hewitt-Nachbin realcompactification of S.

24.4 Another characterization of compactness and of pseudocompactness.

The statements which appear above are mostly in reference to non-compact spaces. The realcompact property provides us with another characterization of the compact property.

Theorem 24.7 A Hausdorff topological space, S, is compact if and only if S is both realcompact and pseudocompact.

Proof: ($\Rightarrow$) If S is compact, then $S = \beta S$; hence every real-valued function on S is bounded. Then, by definition, S is pseudocompact. Since $S \subseteq \upsilon S \subseteq \beta S$, then $S = \upsilon S$, so S is realcompact.

($\Leftarrow$) Suppose S is both pseudocompact and realcompact. We are required to show that S is compact.

Since S is pseudocompact every $f \in C(S)$ is bounded. Then, for such an f,

$$f^{\beta(\omega)}[\beta S] \subseteq \mathrm{cl}_{\mathbb{R}} f[S] \subseteq \mathbb{R}$$

Then

$$\beta S \subseteq \cap\{f^{\beta(\omega)\leftarrow}[\mathbb{R}] : f \in C(S)\} = \upsilon S$$

So $\upsilon S = \beta S$. Since S is realcompact, then $S = \upsilon S$. So $S = \beta S$. So S is compact, as required.

Theorem 24.8 Let S be a non-compact completely regular space. Then the following are equivalent.

(a) The space S is pseudocompact.

(b) The equality, $\beta S = \upsilon S$, holds true.

(c) Every z-ultrafilter is a real z-ultrafilter. That is, every z-ultrafilter satisfies the countable intersection property.

Proof: We are given that S is a non-compact completely regular space.

(a) $\Rightarrow$ (b) We are given that, S is pseudocompact.
We are required to show that $\beta S = \upsilon S$.

Then, for any $f \in C(S)$, $f^{\beta(\omega)} : \beta S \to \omega\mathbb{R}$ maps βS onto $\mathrm{cl}_{\omega\mathbb{R}} f[S] \subseteq \mathrm{cl}_{\mathbb{R}} f[S] \subseteq \mathbb{R}$. Then, for any $f \in C(S)$, we have $\upsilon_f S = \beta S$. So

$$\beta S = \cap\{\upsilon_f S : f \in C(S)\} = \upsilon S$$

as required.

(b) $\Rightarrow$ (a) We are given that, $\beta S = \upsilon S$.
We are required to prove that S is pseudocompact.

Suppose not. Then there is an unbounded function, f, in $C(S)$. Then $f : S \to \mathbb{R}$ extends to $f^{\beta(\omega)} : \beta S \to \omega\mathbb{R}$. If $f^{\beta(\omega)}[\beta S] \subseteq \mathbb{R}$, then $f^{\beta(\omega)}[\beta S]$ is non-compact since it is unbounded. So there is an $x \in \beta S \backslash S$ such that $f^{\beta(\omega)}(x) = \infty$. In such a case, $\beta S \neq \upsilon S$. Contradiction. So S must be pseudocompact.

(b) $\Leftrightarrow$ (c) Given that S is a non-compact completely regular space.

See that $\beta S = \upsilon S$ if and only if every point in βS is a real point. A z-ultrafilter converges to a real point if and only if it is a real z-ultrafilter. So $\beta S = \upsilon S$ if and only if every z-ultrafilter is a real z-ultrafilter. [8]

By the above result, if S is a non-compact pseudocompact space, $\beta S = \upsilon S \neq S$. We can them say with certainty that ...

> "*A non-compact pseudocompact space cannot be realcompact.*"

Furthermore, from Theorem 23.9, we are sure that ...

> "*If a non-compact space, S, is realcompact then βS is not a singular compactification.*"

---

[8] We can also conclude that "...if and only if every z-ultrafilter satisfies the countable intersection property".

24.5 A few examples of realcompact spaces.

We have seen up to now that the simplest examples of non-realcompact spaces are the non-compact pseudocompact spaces.

We will now show that simple examples of realcompact spaces can be found in the class of Lindelöf spaces. In Theorem 16.3, we showed that in any Lindelöf space, S, every filter of *closed* sets in $\mathscr{P}(S)$ which satisfies the "countable intersection property" has a non-empty intersection (in S). In fact, this property characterizes Lindelöf spaces. Now, every real z-ultrafilter in $Z[S]$ is a collection of closed subsets of S which satisfies the CIP. Then, if S is Lindelöf, every real z-ultrafilter in $Z[S]$ has a non-empty intersection. That is, every real z-ultrafilter in a Lindelöf space S converges to a point in S. Then, by Theorem 24.3, Lindelöf spaces are realcompact. We state this as a formal result.

Theorem 24.9 If S is a Lindelöf space, then S is realcompact.

Proof: Given in the paragraph above.

Example 2. By definition 18.9, a σ-compact space is the countable union of compact sets. State conditions under which a σ-compact space is guaranteed to be realcompact.

Solution: In Theorem 18.10 it is shown that in a locally compact Hausdorff space the σ-compact property is equivalent to the Lindelöf property. So any locally compact Hausdorff σ-compact space is realcompact.

Example 3. Show that $\mathbb{N}$ is realcompact.

Solution: Any countable space is the union of singleton sets and hence is σ-compact. Since $\mathbb{N}$ is locally compact Hausdorff and countable $\mathbb{N}$ is realcompact.

Example 4. Show that any *separable metric space* is realcompact.

Solution: By definition, a second countable space is one that has a countable open base. By Theorem 16.3, for metric spaces, S, the following are equivalent: 1) S satisfies the second countable property, 2) S is separable 3) S satisfies the Lindelöf property. By Theorem 24.9, a separable metric space is realcompact.

Example 5. Show that, for $n = 1, 2, 3, \ldots$, every subspace of $\mathbb{R}^n$ equipped with the usual topology is realcompact.

Solution: The Euclidean space $\mathbb{R}^n$ is a finite product of the metric separable space $\mathbb{R}$ and hence is both metrizable and separable (see 7.10 on products of separable spaces and 31.1 on countable products of metrizable spaces). Then it is second countable (see 5.13 where it is shown that metrizable spaces which are separable are also second countable). So every subspace of $\mathbb{R}^n$ is both second countable and metrizable (see the example after definition 5.14 of hereditary property and Theorem 5.15 where we see that both metrizability and second countability are hereditary). In metrizable spaces the Lindelöf property is equivalent to the separable property and the second countable property (see 16.3). Then every subspace of $\mathbb{R}^n$ is Lindelöf. Since Lindelöf spaces are realcompact (by Theorem 24.9), then every subspace of $\mathbb{R}^n$ is realcompact.

Since $\mathbb{R}$ is realcompact then $\upsilon\mathbb{R} = \mathbb{R}$. So, for any point $x \in \beta\mathbb{R}\backslash\mathbb{R}$, there is at least one function f in $C(\mathbb{R})$ such that $f^{\beta(\omega)}(x) = \infty$.

Since any subspace of $\mathbb{R}$ is realcompact, then,

> *... the set of rationals, $\mathbb{Q}$, and the set of irrationals, $\mathbb{J}$, are both realcompact spaces.*

24.6 Properties of realcompact spaces.

We now examine when a particular space inherits the realcompact property from other spaces known to be realcompact. We show, in particular, that the realcompact property is closed under arbitrary intersections and arbitrary products. We also show that a closed subspace inherits the realcompact property from its superset.

Theorem 24.10 Let $\{S_\alpha : \alpha \in I\}$ be a family of non-empty completely regular spaces each of which is realcompact.

(a) If S is a realcompact space which contains each S_α and $T = \cap\{S_\alpha : \alpha \in I\}$ is non-empty in S, then T is a realcompact subspace of S.

(b) The product space, $\prod_{\alpha\in I} S_\alpha$, is realcompact.

Proof: We are given that $\{S_\alpha : \alpha \in I\}$ is a family of completely regular non-empty *realcompact* spaces.

(a) Suppose the S_α's are all subspaces of a realcompact space S and

$$T = \cap\{S_\alpha : \alpha \in I\}$$

is non-empty. We are required to show that T is realcompact. To do this, it suffices to show that $T = \upsilon T$.

Since each S_α is realcompact, $S_\alpha = \upsilon S_\alpha$. Since subspaces of completely regular spaces are completely regular, $T = \cap\{S_\alpha : \alpha \in I\}$ is a completely regular subspace.

We claim that, for each α, $\upsilon T \subseteq \upsilon S_\alpha$. Let i_α be the identity map embedding T into S_α. Then, for each α, i_α extends to a unique ${i_\alpha}^\upsilon : \upsilon T \to \upsilon S_\alpha$ (By Theorem 24.4). Since T is dense in υT, $(i_\alpha|_T)^\upsilon = i_\alpha$ on υT, so $(i_\alpha|_T)^\upsilon$ embeds υT into υS_α, establishing the claim.

Since $(i_\alpha|_T)^\upsilon[\upsilon T] = \upsilon T \subseteq \upsilon S_\alpha = S_\alpha$ for all $\alpha \in I$, then $\upsilon T \subseteq \cap\{S_\alpha : \alpha \in I\} = T$.
Given that $T \subseteq \upsilon T$, we can only conclude that $T = \upsilon T$ and so T is realcompact.

(b) Suppose $S = \prod_{\alpha \in I} S_\alpha$ where each S_α is realcompact. We know that the product of completely regular spaces is completely regular. For $\gamma \in I$, the projection map, $\pi_\gamma : S \to S_\gamma$ is continuous and hence extends continuously to

$$\pi_\gamma^\upsilon : \upsilon S \to \upsilon S_\gamma$$

where $S_\gamma = \upsilon S_\gamma$.

Let $\mathscr{G} = \{\pi_\alpha : \alpha \in I\}$ where $\pi_\alpha : S \to S_\alpha$. Then the evaluation map, $e_\mathscr{G} : S \to \prod_{\alpha \in I} S_\alpha$ extends to ${e_\mathscr{G}}^\upsilon : \upsilon S \to \prod_{\alpha \in I} S_\alpha = S$ a function which embeds υS into S.

Then $\upsilon S \subseteq S$. Since $S \subseteq \upsilon S$, we conclude that $S = \upsilon S$ and so S is realcompact.

The subspace, T, of a realcompact space, S, need not, generally, be realcompact. We now show that, in the particular case where T is closed in S, it is.

Theorem 24.11 If F is a closed non-empty subspace of a realcompact space S, then F is realcompact.

Proof : We are given that F is closed in realcompact S. Then $S = \upsilon S$ hence

$$\mathrm{cl}_{\upsilon S} F = \mathrm{cl}_S F = F$$

(since F is closed in S). We are required to show that $F = \upsilon F$.

Let $i : F \to S$ denote the inclusion map embedding F into S. Then (by Theorem 24.4) $i : F \to F \subseteq S$ extends continuously to $i^\upsilon : \upsilon F \to \upsilon S$ where

$$i^\upsilon[\upsilon F] = \upsilon F \subseteq \upsilon S = S$$

Now F is dense in υF hence

$$\upsilon F \subseteq \mathrm{cl}_{\upsilon S} F = \mathrm{cl}_S F = F$$

Then $\upsilon F \subseteq F$. So $F = \upsilon F$. We can then conclude that F is a realcompact subset of S.

Lemma 24.12 Closed subspaces of a product of $\mathbb{R}$ are realcompact.

Proof : Let F be a closed subset of the product space, $S = \prod_{\alpha \in I} \mathbb{R}^\alpha$. Then, by Theorem 24.10 part (b), S is realcompact. By, Theorem 24.11, F is realcompact.

For the following lemma note that υS is completely regular (given that $\upsilon S \subseteq \beta S$). Then $C(\upsilon S)$ separates points and closed sets of υS. Then the evaluation function $e^\upsilon_{C(S)}$ maps υS homeomorphically into $\prod_{f \in C(S)} \mathbb{R}_f$ (as argued in the embedding theorem I).

Lemma 24.13 Suppose S is completely regular. Then the function

$$e^\upsilon{}_{C(S)} : \upsilon S \to \prod_{f \in C(S)} \mathbb{R}_f$$

maps υS homeomorphically onto a closed subset of $\prod_{f \in C(S)} \mathbb{R}_f$. Hence, if S is realcompact, $e_{C(S)}[S]$ maps S homeomorphically onto a closed subset of $\prod_{f \in C(S)} \mathbb{R}_f$.

Proof: We are given that S is completely regular. For each $x \in \beta S$, $f^{\beta(\omega)}(x) \in \omega\mathbb{R}_f$. Then the evaluation map $e^{\beta(\omega)}_{C(S,\omega\mathbb{R})}$ maps βS onto a compact subspace of the product space $\prod_{f\in C(S,\omega\mathbb{R})} \omega\mathbb{R}_f$ as follows:

$$e^{\beta(\omega)}_{C(S,\omega\mathbb{R})}(x) \ = < f^{\beta(\omega)}(x) >_{f\in C(S,\omega\mathbb{R})} \in \ \prod_{f\in C(S,\omega\mathbb{R})}\omega\mathbb{R}_f$$

Recall that S is densely contained into its realcompactification υS. Then every (real-valued) function f in $C(S)$ extends continuously to (a real-valued function) $f^\upsilon = f^{\beta(\omega)}|_\upsilon \in C(\upsilon S)$ where $f^\upsilon[\upsilon S] = f^{\beta(\omega)}[\upsilon S] \subseteq \mathbb{R}_f \subseteq \omega\mathbb{R}_f$.

The evaluation map $e_{C(S)} : S \to \prod_{f\in C(S)} \mathbb{R}_f$ then extends continuously to the evaluation map $e^\upsilon_{C(S)}$ mapping υS into $\prod_{f\in C(S)}\mathbb{R}_f \subseteq \prod_{f\in C(S)}\omega\mathbb{R}_f$. It is defined as

$$e^\upsilon_{C(S)}(x) \ = < f^\upsilon(x) >_{f\in C(S)} \in \ \prod_{f\in C(S)}\mathbb{R}_f \ \subseteq \ \prod_{f\in C(S,\ \omega\mathbb{R})}\omega\mathbb{R}_f$$

Claim: We claim that

$$e^\upsilon_{C(S)}[\upsilon S] \ = \ e^{\beta(\omega)}_{C(S,\ \omega\mathbb{R})}[\beta S] \ \cap \ \prod_{f\in C(S)}\mathbb{R}_f$$

Proof of claim. See that,

$$\begin{aligned} e^\upsilon_{C(S)}[\upsilon S] &= e^{\beta(\omega)}_{C(S,\ \omega\mathbb{R})}[\beta S \cap \upsilon S] \\ &= e^{\beta(\omega)}_{C(S,\ \omega\mathbb{R})}[\beta S] \ \cap \ e^{\beta(\omega)}_{C(S,\ \omega\mathbb{R})}[\upsilon S] \quad \text{(Since } e^{\beta(\omega)}_{C(S,\ \omega\mathbb{R})} \text{ is one-to-one on } \upsilon S\text{)} \\ &= e^{\beta(\omega)}_{C(S,\ \omega\mathbb{R})}[\beta S] \ \cap \ e^{\beta(\omega)}_{C(S,\ \omega\mathbb{R})}\left[{e^{\beta(\omega)}_{C(S,\ \omega\mathbb{R})}}^{\leftarrow}\left[\ \prod_{f\in C(S)}\mathbb{R}_f\right]\right] \\ &\subseteq e^{\beta(\omega)}_{C(S,\ \omega\mathbb{R})}[\beta S] \ \cap \ \prod_{f\in C(S)}\mathbb{R}_f \quad \text{(A closed subset of } \textstyle\prod_{f\in C(S)}\mathbb{R}_f\text{)} \end{aligned}$$

So LHS $\subseteq$ RHS.

To prove RHS $\subseteq$ LHS, suppose

$$< f^{\beta(\omega)}(x) >_{f\in C(S,\omega\mathbb{R})} \ \in \ e^{\beta(\omega)}_{C(S,\ \omega\mathbb{R})}[\beta S] \ \cap \ \prod_{f\in C(S)}\mathbb{R}_f$$

Then $f^{\beta(\omega)}(x)$ is a real number for all $f \in C(S,\omega\mathbb{R})$. Then x is a real point of S. That is, $x \in \upsilon S$.

Then $f^{\beta(\omega)}(x) \in f^{\beta(\omega)}[\upsilon S]$ for all $f \in C(S,\omega\mathbb{R})$.

That is,

$$< f^{\beta(\omega)}(x) >_{f\in C(S,\omega\mathbb{R})} \ \in \ < f^{\beta(\omega)}[\upsilon S] >_{f\in C(S,\omega\mathbb{R})}$$

Since $< f^{\beta(\omega)}[\upsilon S] >_{f \in C(S, \omega\mathbb{R})} \; = \; e^{\upsilon}_{C(S)}[\upsilon S]$, then

$$< f^{\beta(\omega)}(x) >_{f \in C(S, \omega\mathbb{R})} \;\; \in \;\; e^{\upsilon}_{C(S)}[\upsilon S]$$

So RHS $\subseteq$ LHS.

So

$$e^{\upsilon}_{C(S)}[\upsilon S] \quad = \quad e^{\beta(\omega)}_{C(S,\ \omega\mathbb{R})}[\beta S] \;\cap\; \textstyle\prod_{f \in C(S)} \mathbb{R}_f$$

as claimed.

We conclude So $e^{\upsilon}_{C(S)}$ maps υS homeomorphically onto the closed subset

$$e^{\beta(\omega)}_{C(S,\ \omega\mathbb{R})}[\beta S] \;\cap\; \textstyle\prod_{f \in C(S)} \mathbb{R}_f$$

of $\prod_{f \in C(S)} \mathbb{R}_f$. As required.

Finally if S is realcompact then $S = \upsilon S$, so $e^{\upsilon}_{C(S)}$ maps S homeomorphically onto the closed subset

$$e^{\beta(\omega)}_{C(S,\ \omega\mathbb{R})}[\beta S] \;\cap\; \textstyle\prod_{f \in C(S)} \mathbb{R}_f$$

of $\prod_{f \in C(S)} \mathbb{R}_f$.

This completes the proof of the lemma.

Theorem 24.14 The realcompact spaces are precisely those spaces which are homeomorphic to a closed subspace of a power of $\mathbb{R}$.

Proof: We have already shown that the closed subspaces of a power of $\mathbb{R}$ are realcompact.

Suppose, on the other hand, that S is a realcompact space. That is, suppose that $S = \upsilon S$.

We have shown in the lemma above that $e^{\upsilon}{}_{C(S)}$ maps, υS, homeomorphically onto a closed subset of a power of $\mathbb{R}$. Since $S = \upsilon S$ then S is homeomorphic to a closed subset of a power of $\mathbb{R}$.

We have seen that any closed subset of a realcompact space is realcompact. But as shown below, a subset of a realcompact space need not, necessarily, be closed for it to be realcompact. Cozero-sets also satisfy this property.

Theorem 24.15 A cozero-set in a realcompact space is itself realcompact.

Proof: Suppose S is realcompact and U is a cozero-set in S.

To show that U is realcompact, it will suffice to show that, for every $p \in \beta U \setminus U$ there is a βU-zero-set, Z, such that $p \in Z \subseteq \beta U \setminus U$ and invoke part (c) of Theorem 24.3.

See that the identity map $i : U \to \mathrm{cl}_{\beta S} U$ extends continuously to $i^* : \beta U \to \mathrm{cl}_{\beta S} U$. Let $p \in \beta U \setminus U$ and $p^* = i^*(p)$. Then $p^* \in \mathrm{cl}_{\beta S} U \setminus U$ (see example on page 135).

Step #1. We claim there is a $\mathrm{cl}_{\beta S} U$-zero-set, Z^*, such that $p^* \in Z^* \subseteq \mathrm{cl}_{\beta S} U \setminus U$.

Proof of claim: We know that $p^* \in \mathrm{cl}_{\beta S} U \setminus U$. We consider two cases: Either $p^* \in \mathrm{cl}_{\beta S} U \cap \beta S \setminus S$ or $p^* \in \mathrm{cl}_{\beta S} U \cap S$.

Case 1. Suppose $p^* \in \mathrm{cl}_{\beta S} U \cap \beta S \setminus S$. Since S is realcompact, by part (c) of Theorem 24.3, there exists a βS-zero-set, Z, such that $p^* \in Z \subseteq \beta S \setminus S$. Then $p^* \in Z^* = Z \cap \mathrm{cl}_{\beta S} U \subseteq \mathrm{cl}_{\beta S} U \setminus U$. The claim is established for the Case 1.

Case 2. Suppose $p^* \in \mathrm{cl}_{\beta S} U \cap S$. We are given that

$$p^* \in \mathrm{cl}_S U \setminus U$$

where U is a cozero-set of S. Since $S \setminus U$ is a zero-set, then $S \setminus U$ is a G_δ, so U is an F_σ (a countable union of closed subsets of S). Say $U = \cup\{F_n : n \in \mathbb{N}\}$ where F_n is a closed subset of S for all n. Since each F_n is closed, for each n, there is an S-zero-set, Z_n, such that

$$p^* \in Z_n \subseteq S \setminus F_n$$

Then $p^* \in \cap\{Z_n : n \in \mathbb{N}\} = Z$, itself an S-zero-set (see page 250). Note that Z does not intersect U. There is a βS-zero-set, $Z^\beta = \mathrm{cl}_{\beta S} Z$ such that $Z = Z^\beta \cap S$. Let

$$Z^* = Z^\beta \cap \mathrm{cl}_{\beta S} U$$

a $\text{cl}_{\beta S}U$-zero-set such that $p^* \in Z^* \subseteq \text{cl}_{\beta S}U \backslash U$. The claim is established for the Case 2.

Step #2. We have thus shown that, for both cases

$$p^* \in Z^* \subseteq \text{cl}_{\beta S}U \backslash U$$

where Z^* is a $\text{cl}_{\beta S}S$-zero-set. Say $Z^* = Z^*(f)$. Then, since $f^{\beta}(i^{*\leftarrow}(x)) = 0$ if and only if $x \in Z^*(f)$, then $p \in i^{*\leftarrow}[Z^*]$ is a βS-zero-set contained in $\beta U \backslash U$. So, part (c) of Theorem 24.3, U is realcompact.

Theorem 24.16 A space S is pseudocompact if and only if every countable C-embedded subset of S is compact.

Proof: ($\Rightarrow$) Suppose S is a pseudocompact space. Let $T = \{x_i : i \in \mathbb{N}\}$ be a countable C-embedded subset of S. We are required to show that T is compact.

It suffices to show that T is both realcompact and pseudocompact. Suppose T is not pseudocompact. Then there is a continuous unbounded real-valued function $f : T \to \mathbb{R}$ on T. Since T is C-embedded in S then f extends to an unbounded function $f^* : S \to \mathbb{R}$. This contradicts the pseudocompactness of S. Then T is pseudocompact.

Since T is countably infinite it is Lindelöf. By Theorem 24.9, Lindelöf spaces are realcompact; so T is realcompact. By Theorem 24.9, since T is both realcompact and pseudocompact it is compact.

($\Leftarrow$) Suppose every countable C-embedded subset of S is compact. A C-embedded copy of $\mathbb{N}$ in S is not a compact subset of S. Since T is compact it cannot contain a C-embedded copy of $\mathbb{N}$. Then by Theorem 23.4 S is pseudocompact.

Theorem 24.17 If a completely regular space S is pseudocompact then every countable zero set in S is compact.

Proof: Suppose S is completely regular and pseudocompact. Suppose Z is a countable zero set of S. We are required to show that Z is a compact set.

To show that Z is compact it is sufficient to show that each filter of closed subsets of Z has non-empty intersection in Z. Let $\mathscr{D} = \{D_i :$

$i \in I\}$ be a filter of closed subsets of Z. Then $\mathscr{D}$ satisfies the *finite intersection property.* Since Z is closed in S, each D_i is closed in S.

We claim that pseudocompactness and complete regularity of S along with countability of Z are sufficient conditions to guarantee that the filter $\mathscr{D}$ has non-empty intersection in Z.

See that

$$\begin{aligned} D_0 \cap D_1 &= Z\backslash F_0 \text{ for some } \; F_0 \text{ open in } Z \\ D_0 \cap D_1 \cap D_2 &= Z\backslash (F_0 \cup F_1) \text{ for some } \; F_1 \text{ open in } Z \\ &\vdots \\ \cap\{D_i : i = 0 \text{ to } k\} &= Z\backslash \cup\{F_i : i = 0 \text{ to } k-1\} \text{ for some } \; F_{k-1} \text{ open in } Z \end{aligned}$$

We see that

$$\begin{aligned} \cap\{D_i : i = 0 \text{ to } k\} &= Z\backslash \cup\{F_i : i = 0 \text{ to } k-1\} \\ &= [\cap\{Z\backslash F_i : i \;=\; 0 \text{ to } k-1\}] \;\cap\; Z \end{aligned}$$

See that for each i, $Z\backslash F_i \cap Z$ is closed in S. Since S is completely regular, for each $x_j \in F_i \cap Z$ there exists $Z_j \in Z[S]$ such that $(Z\backslash F_i \cap Z) \subseteq Z_j$, $x_j \notin Z_j$. Then, for each i,

$$Z\backslash F_i \cap Z = \cap\{Z_j \in Z[S] : x_j \in F_i \cap Z\}$$

equal to a zero-set $Z_i \in Z[S]$.

So

$$\cap\{D_i : i = 0 \text{ to } k\} = [\cap\{Z_i \in Z[S] : i \;=\; 0 \text{ to } k-1\}] \;\cap\; Z$$

a zero-set in S.

Also see that, for each k,

$$\cap\{D_i : i = 0 \text{ to } k\} \;\subseteq\; Z_i \cap Z, \quad \text{for } i = 0 \text{ to } k-1$$

Then $\{Z_i \cap Z \in Z[S] : i = 0 \text{ to } k-1\}$ is a z-filter in S.
We can then say that,

$$\cap\{D_i : i \in I\} \;=\; Z\backslash \cup\{F_i : i \in \mathbb{N}\} \;=\; \cap\{Z_i \in Z[S] : i \in \mathbb{N}\} \;\cap\; Z$$

Since S is pseudocompact, then by Theorem 23.4, the z-filter, $\{Z_i \cap Z : i \in \mathbb{N}\}$, satisfies the countable intersection property. Then $\cap\{Z_i \in Z[S] : i \in \mathbb{N}\} \;\cap\; Z$ is non-empty. So the filter $\mathscr{D} = \{D_i : i \in I\}$ of closed subsets of Z has non-empty intersection. Since this holds true for an arbitrary choice of filters of closed sets in Z, it holds true for all filters. We can then conclude that Z is compact, as required.

Example 6. See the topological ψ-space described in the example on page 111. In the example on page 399, we show that the ψ-space is pseudocompact. In the example on page 421 we showed that the ψ-space is Hausdorff and locally compact, hence is completely regular. We also show that a basic open neighborhood of $D \in \mathscr{M}$, of the form $\{D\} \cup D \backslash F$, is compact, hence the space is zero-dimensional. Show that the finite sets in $\mathscr{M}$ are zero-sets, but no countably infinite subset of $\mathscr{M}$ can be a zero-set in ψ.

Solution: Every subset in ψ of the form $\psi \backslash F$ where $F \subseteq \mathbb{N}$ is a zero-set in ψ. Since $\mathscr{M} = \cap\{\psi \backslash F : F \subseteq \mathbb{N}\}$ and countable intersections of zero-sets are known to be zero-sets, $\mathscr{M}$ is a zero-set in ψ. Since the basic open neighborhoods $\{D\} \cup D \backslash F$ of a point D in $\mathscr{M}$ were shown to be clopen then these neighborhoods are zero-sets in ψ. Then the intersection, $(\{D\} \cup D \backslash F) \cap \mathscr{M} = \{D\}$ of two zero-sets is a zero-set. So finite subsets of $\mathscr{M}$ are zero-sets.

Suppose F is a countably infinite subset of $\mathscr{M}$. Since F is discrete in the closed subset $\mathscr{M}$ of ψ it cannot be compact. Since ψ is pseudocompact, by the Theorem 24.17 above, countable zero-sets must be compact. So $\mathscr{M}$ cannot contain countably infinite zero-sets.

24.7 Example of a non-realcompact non-pseudocompact space.

The above few examples may lead the reader to suspect that non-realcompact non-pseudocompact spaces are harder to come by. To come up with such an example, one might consider a product of a realcompact space with a pseudocompact space.

Example 7. Recall that ω_1 denotes the first uncountable ordinal, while ω_0 denotes the first countable infinite ordinal. Consider the spaces $[0, \omega_1)$ and $[0, \omega_0)$ both be equipped with the ordinal topology, and let

$$S = [0, \omega_1) \times [0, \omega_0)$$

be the corresponding product space.[11] This is a slight variation of the Tychonoff plank $[0, \omega_1] \times [0, \omega_0]$.

We first verify that $[0, \omega_1)$ *is not realcompact.* In the example on page 368, it is shown that $[0, \omega_1)$ is countably compact. By Theorem 15.9, since $[0, \omega_1)$ is countably compact, for any $f \in C(X)$, $f[[0, \omega_1)]$ is

[11] In Theorem 21.18, it is shown that $\beta X = [0, \omega_1] = \omega X$.

compact in $\mathbb{R}$. This implies that $[0, \omega_1)$ is pseudocompact. But the space $[0, \omega_1)$ is non-compact since the open cover $\{[0, \gamma) : \gamma \in S\}$ has no finite subcover. By Theorem 24.7, pseudocompact spaces which are realcompact must be compact. So

$$[0, \omega_1) \text{ is not realcompact}$$

We claim that $S = [0, \omega_1) \times [0, \omega_0)$ is not realcompact. Suppose S is realcompact. Note that the subset $[0, \omega_1) \times \{0\}$ is closed in S. The Theorem 24.11 states that closed subspaces of realcompact spaces are realcompact. But we just showed that $[0, \omega_1)$ is not realcompact, a contradiction. Then S cannot be realcompact, as claimed.

We now claim that the non-realcompact space is not pseudocompact. To show this it suffices to produce an unbounded real-valued continuous function. For $n \in \omega_0$, let $A_n = [0, \omega_1) \times \{n\}$. Then, for each n, A_n is clopen in S. The function $f : S \to \mathbb{N}$ defined as $f[A_n] = n$ is continuous, real-valued and unbounded on S. Then S cannot be pseudocompact, as claimed.

Note that, since the space $S = [0, \omega_1) \times \{0\}$ is not pseudocompact, then, by Theorem 24.7, $\beta S \neq \upsilon S$. Since S has been shown to be non-realcompact, $S \neq \upsilon S$. So, for $S = [0, \omega_1) \times [0, \omega_0)$, we have

$$S \subset \upsilon S \subset \beta S$$

Concepts review.

1. If f is a function in $C(S)$ define the set, $\upsilon_f S$, of all the real points of f.
2. Define the realcompactification υS of a space S in terms of all the sets $\upsilon_f S$.
3. What does it mean to say that the space, S, is realcompact? Is $\mathbb{R}$ realcompact?
4. Define a real z-ultrafilter in $Z[S]$.

5. Give a characterization of the realcompactness property in terms of zero-sets in $Z[\beta S]$.

6. Give a characterization of the realcompactness property in terms of z-ultrafilters of zero-sets in $Z[S]$.

7. Let $f : S \to T$ be a continuous function mapping the completely regular space S into the completely regular space T what can be said about a particular continuous extension of f to f^{υ}?

8. What does it mean to say that $f \in C(S)$ is C-embedded in υS?

9. Is υS a realcompact space?

10. Describe $\mathrm{cl}_{\beta S}\upsilon S$ and $\beta(\upsilon S)$.

11. What can we say about a space that is both realcompact and pseudo-compact?

12. Complete the statement "$\beta S = \upsilon S$ if and only if . . . ".

13. What can we say about Lindelöf spaces in the context discussed in this chapter?

14. Are the subspaces $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{J}$ of $\mathbb{R}$ realcompact?

15. What can we say about arbitrary intersections of realcompact space?

16. What can we say about arbitrary products of realcompact space?

17. What kind of subsets of a realcompact space are guaranteed to be realcompact?

18. Give an example of a realcompact space.

19. State a characterization of realcompact spaces in terms of subsets of a product of $\mathbb{R}$.

20. Give an example of a space which is not realcompact.