15. Characterize the *locally compact* property of a space $S$ in terms of its compactification $\alpha S$ of $S$.

16. Suppose $T$ is a non-empty subset of a completely regular space $S$. What property must $T$ satisfy in relation to $S$ if $\mathrm{cl}_{\beta S} T = \beta T$?

17. We know the cardinalities $|\mathbb{N}| = |\mathbb{Q}| = \aleph_0$ and $|\mathbb{R}| = c$. What are the cardinalities of $\beta\mathbb{N}$, $\beta\mathbb{Q}$ and $\beta\mathbb{R}$?

# 22 / Singular sets and singular compactifications.

**Abstract**. *In this chapter, we introduce an alternate method to construct a compactification of a locally compact Hausdorff space. We define the notion of the singular set, $S(f)$, of a function, $f : S \to T$. We will show that we can always use $S(f)$ to construct a compactification, $\alpha S = S \cup S(f)$, by applying a suitable topology on $\alpha S$. If the singular set, $S(f)$, contains the image, $f[S]$, of $f$, we refer to $f$ as a singular map. When $f$ is singular the resulting compactification, $S \cup_f S(f)$, is called a singular compactification.*

## 22.1 Singular sets and functions: definitions.

We begin by formally defining the "singular set" of a continuous function on a locally compact non-compact Hausdorff space. We also define what is meant when a function is referred to as being a "singular map".

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**Definition 22.1** Let $(S, \tau)$ be a locally compact non-compact Hausdorff topological space and $f : S \to T$ be a continuous function mapping $S$ into some *compact* space $T$. We define the *singular set,* $S(f)$, of $f$ as follows:

$$S(f) = \{x \in \mathrm{cl}_T f[S] : \mathrm{cl}_S f^{\leftarrow}[U] \text{ is not compact } \forall\ T\text{-open nbhd, } U, \text{ of } x\}$$

If $S(f) = \mathrm{cl}_T f[S]$, then $f : S \to T$ is said to be a *singular function* or *singular map.*

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We make a few observation about the singular set $S(f)$ of a function $f$.

1. *If $S$ is a non-compact locally compact Hausdorff space, the singular set $S(f)$ is never empty in the compact codomain, $T$.* This is so, whether $f$ is a singular map or not. To see this, recall that, by Theorem 21.5, $f : S \to T$, extends to $f^{\beta(T)} : \beta S \to T$. Suppose $u \in \beta S \setminus S$ and
$$x = f^{\beta(T)}(u) \in f^{\beta(T)}[\beta S \setminus S]$$
If $U$ is an open neighborhood of $x$ in $T$, then $u \in f^{\beta(T)\leftarrow}[U]$ is open in $\beta S$. If $\mathrm{cl}_S f^{\leftarrow}[U]$ is a compact subset of $S$, then $f^{\beta(T)\leftarrow}[U] \setminus \mathrm{cl}_S f^{\leftarrow}[U]$ is an open neighborhood of $u$ contained in $\beta S \setminus S$, a contradiction. So $\mathrm{cl}_S f^{\leftarrow}[U]$ is not compact. By definition, $x \in S(f)$. So $S(f)$ is non-empty.

2. *The singular set $S(f)$ is always closed, and hence compact, in $T$.* This is so whether $f$ is a singular map or not. To see this, suppose $x \in T\backslash S(f)$. Then there exists an open neighborhood $U$ of $x$ in $T$, such that $\mathrm{cl}_S f^{\leftarrow}[U]$ is compact in $S$. Then every point $p \in U$ also belongs to $T\backslash S(f)$. So $x \in U \subseteq T\backslash S(f)$. Hence $S(f)$ is a closed (and so is a compact) subset of the compact space $T$.
3. If $f : S \to T$ is a *singular map*, then $f[S]$ is a dense subset of $S(f)$ (since, by definition of singular function, $S(f) = \mathrm{cl}_T f[S]$).

## 22.2 Singular compactifications: Definitions.

We now show how the singular set, $S(f)$, of a function, $f : S \to T$, can be used as the outgrowth of a compactification of $S$.

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**Definition 22.2** Let $(S, \tau)$ be a locally compact non-compact Hausdorff topological space and $f : S \to T$ be a continuous function mapping $S$ into a compact space $T$. Then $S(f)$ denotes its singular set. If $f$ is a singular map, then, by definition, $S(f) = \mathrm{cl}_T f[S] \subseteq T$. We construct a new set by adjoining $S(f)$ to $S$ to obtain a larger set,

$$\gamma S = S \cup_f S(f)$$

The basic open neighborhoods, $\mathscr{B}_1$, of points in $S$ will be the same as the ones in $S$ when viewed as a topological space on its own. The locally compact property guarantees that $S$ is open in $\gamma S$.[1] If $x \in S(f)$, $F$ is a compact subset of $S$ and $U$ an open neighborhood of $x$ in $S(f)$, we define

$$B_x = U \cup f^{\leftarrow}[U]\backslash F$$

as a basic open neighborhood of $x$. Let $\mathscr{B}_2 = \{B_x : x \in S(f)\}$. Let $\mathscr{B} = \mathscr{B}_1 \cup \mathscr{B}_2$. This defines a base for a topology on $\gamma S$ which is easily seen to be a Hausdorff compactification of $S$. We will refer to $\gamma S$ as the *singular compactification* induced by the singular map $f : S \to T$.

---

[1] Remember that, if $S$ is locally compact Hausdorff, then $S$ is open in any compactification of $S$.

*A few remarks.*

*On compactness of* $S \cup_f S(f)$. We first confirm that $S \cup_f S(f)$ is indeed a compact set. Let $\mathscr{U}$ be an open cover of $S \cup_f S(f)$ with elements of $\mathscr{B}$. Let $\mathscr{U}_1$ be a subset of $\mathscr{U}$ which covers $S(f)$. Since $S(f)$ is compact, $\mathscr{U}_1$ contains a finite subcover $\{U_i \cup f^{\leftarrow}[U_i] : i \in F\}$ of $S(f)$. Since $f$ is a singular map, $f[S] \subseteq S(f)$; then $\cup\{f^{\leftarrow}[U_i] : i \in F\}$ covers $S$. So $\mathscr{U}_1$ is a finite subcover of $S \cup_f S(f)$. So $S \cup_f S(f)$ is indeed compact, as claimed.

*On locally compact property of* $S$. Secondly, note that, the locally compact property on $S$, will guarantee that $\alpha S = S \cup_f S(f)$ is a Hausdorff compactification which densely contains $S$ as an open subspace. To see this, given distinct points $x$ and $y$ in $S(f)$ and distinct open neighborhoods, $U$ and $V$ of $x$ and $y$ in $S(f)$, $U \cup f^{\leftarrow}[U]$ and $V \cup_f f^{\leftarrow}[V]$ form disjoint open neighborhoods of $x$ and $y$ in $S \cup_f S(f)$.

But what if $x \in S(f)$ and $y \in S$? The locally compact property guarantees that $y$ has a compact neighborhood, $\mathrm{cl}_S V$ in $S$, such that $(U \cup f^{\leftarrow}[U] \backslash \mathrm{cl}_S V) \cap \mathrm{cl}_S V = \varnothing$. Without local compactness of $S$ this may not be possible. If we discuss a compactification $S \cup_f S(f)$ where $S$ is not locally compact, we must keep in mind that $S \cup_f S(f)$ may not be a Hausdorff compactification. This is why, we don't normally discuss the singular compactification of a space such as the non-locally compact space, $\mathbb{Q}$, of rationals, without commenting on whether this aspect is relevant and taken into account.

At first, the definition of a singular compactification is not a simple one to visualize. It invites many questions. For example, if given a singular compactification $\gamma S$, one may want to know which function induces it. Can there be more than one such function which induces it? Are there compactifications which are non-singular?

To help us answer such questions it will be helpful to find simpler characterizations of a singular set of a function and the associated singular compactification.

*Retractions and retracts.* We remind the reader that, for a topological space $S$,

> *. . . if $A \subset S$ and $r : S \to A$ is a continuous function which fixes the points of $A$, then $r$ is referred to as a "retraction" of $S$ onto $A$. In such a case, $A$ is called a "retract" of $S$.*

We will see that those compactifications, $\alpha S$, whose outgrowth, $\alpha S \backslash S$, is a retract of $\alpha S$ characterize singular compactifications.

---

**Theorem 22.3** Let $S$ be locally compact and Hausdorff. Let $\alpha S$ be a Hausdorff compactification of $S$. Then $\alpha S$ is a singular compactification of $S$ if and only if $\alpha S \backslash S$ is a retract of $\alpha S$.

*Proof*: We are given that $\alpha S$ is a Hausdorff compactification of $S$.

( $\Rightarrow$ ) Suppose $\alpha S = S \cup_f S(f)$ is a singular compactification of $S$ induced by the continuous function,

$$f : S \quad \rightarrow \quad \mathrm{cl}_T f[S] = S(f) = \alpha S \setminus S$$

We are required to show that $\alpha S \setminus S$ is a retract of $\alpha S$.

By definition, $f[S]$ is dense in $S(f)$. Let

$$f^\alpha : \alpha S \rightarrow S(f)$$

be a function *which agrees with $f$ on $S$ and fixes the points of $S(f)$.*

We claim, that $f^\alpha$ is continuous on $\alpha S$: Let $x \in S(f)$ and $U$ be an open neighborhood of $x$. Then $f^{\alpha\leftarrow}[U] = U \cup f^{\leftarrow}[U]$, by definition of a basic open neighborhood in $\alpha S$. So $f^\alpha$ is continuous on $\alpha S$, as claimed. So $f^\alpha : S \cup_f S(f) \rightarrow S(f)$ is a retraction of $\alpha S$ onto $S(f)$.

Hence $\alpha S \setminus S$ is a retract of $\alpha S$, as required.

( $\Leftarrow$ ) Suppose $\alpha S \setminus S$ is a retract of $\alpha S$. We are required to show that $\alpha S$ is a singular compactification.

By hypothesis, there is a continuous function $r : \alpha S \rightarrow \alpha S \setminus S$ which fixes the points of $\alpha S \setminus S$. By definition of retract, $r[S] \subseteq \alpha S \setminus S$.

We claim that $r|_S$ is a singular function on $S$: Let $x \in \alpha S \setminus S$ and $U$ be an open neighborhood of $x$ in $\alpha S \setminus S$. Now, $U$ must intersect $r|_S[S]$, for if not, $r^{\leftarrow}[U] = U \subseteq \alpha S \setminus S$ which is not open in $\alpha S$. So $r^{\leftarrow}[U] = U \cup (r^{\leftarrow}[U] \cap S)$. Then, since $\mathrm{cl}_S r|_S{}^{\leftarrow}[S]$ is not compact in $S$, then $\alpha S \setminus S = S(r|_S)$ is the singular set of $r|_S$, and so

$$\alpha S = S \cup_r S(r|_S)$$

a singular compactification induced by the map $r|_S$. So

$$r|_S : S \rightarrow \alpha S \backslash S$$

is a singular map, as claimed.

We now have another way of recognizing a singular compactification:

> *Singular compactifications are precisely those compactifications, $\alpha S$, whose outgrowth, $\alpha S\backslash S$, is a retract of the whole space.*

The above theorem also confirms that

> *... given any singular compactification $\alpha S = S \cup_f S(f)$ induced by a singular map, $f : S \to S(f)$, the function $f$ extends continuously to the function $f^\alpha : \alpha S \to S(f)$.*

In the following example, we verify that, given a singular compactification of $S$, then every compactification "below" it in the partially ordered family of all compactifications will also be a singular compactification.

*Example* 1. Show that, if $\alpha S$ is a singular compactification and $\gamma S$ is another compactification such that $\gamma S \preceq \alpha S$, then $\gamma S$ is also a singular compactification.

*Solution*: Suppose $\alpha S$ is a singular compactification and $\gamma S$ is another compactification such that $\gamma S \preceq \alpha S$. Then $\alpha S = S \cup_f S(f)$ where $f : S \to T$ is continuous and $f[S] \subseteq S(f) = \text{cl}_T f[S]$. Also, there exists $\pi_{\alpha\to\gamma} : \alpha S \to \gamma S$ such that $\pi_{\alpha\to\gamma}$ is continuous and onto and fixes the points of $S$. If $g = \pi_{\alpha\to\gamma}\circ f$, then

$$\begin{aligned}
\text{cl}_{\gamma S} g[S] &= \text{cl}_{\gamma S}(\pi_{\alpha\to\gamma}\circ f)[S] \\
&= \pi_{\alpha\to\gamma}[\text{cl}_{\alpha S}(f[S])] \\
&= \pi_{\alpha\to\gamma}[\alpha S\backslash S] \\
&= \gamma S\backslash S
\end{aligned}$$

If $U$ is an open subset of $\gamma S\backslash S$

$$\begin{aligned}
\text{cl}_S g^{\leftarrow}[U] &= \text{cl}_S(\pi_{\alpha\to\gamma}\circ f)^{\leftarrow}[U] \\
&= \text{cl}_S f^{\leftarrow}(\pi^{\leftarrow}_{\alpha\to\gamma}[U]) \\
&= \text{cl}_S f^{\leftarrow}[V] \quad \text{(}V\text{ open in }\alpha S\backslash S\text{)} \\
&\quad \text{a non-compact set in } S.
\end{aligned}$$

So $\gamma S\backslash S = S(g)$. This means that $\gamma S\backslash S$ is a retract of $\gamma S$ and so $\gamma S$ is a singular compactification. We are done with the solution.

Does every locally compact Hausdorff space, $S$, have at least one singular compactification? Well, we know that such a space $S$, has a one-point compactification, $\omega S = S \cup \{\omega\}$. Consider the constant function $r : \omega S \to \{\omega\}$. It is a retraction on $\omega S$. So $r|_S : S \to \{\omega\}$ is a singular map on $S$. So we can answer this question in the affirmative.

## 22.3 Compactifications induced by non-singular functions.

Suppose $(S, \tau)$ is a topological space and $f : S \to T$ is a continuous function mapping a non-compact locally compact Hausdorff space $S$ into a compact space $T$. For what follows, $f$ *may or may not* be a singular map.

*Compactifications induced by non-singular, $f$.*

Suppose $S$ is locally compact Hausdorff and $f$ is *not* a singular map on $S$. Recall that

$$S(f) = \{x \in \mathrm{cl}_T f[S] : \mathrm{cl}_S f^{\leftarrow}[U] \text{ is not cpct } \forall\ T\text{-open nbhd, } U, \text{ of } x\}$$

Since $f$ is not singular, then $S(f)$ does not satisfy the condition, $f[S] \subseteq S(f)$, required to construct a singular compactification. Despite this, we can still **adjoin** the singular set $S(f)$ to $S$ to form a larger set,

$$K = S \cup^* S(f)$$

*Important note*: The symbol $\cup^*$ is not to be interpreted as "$S$ union $S(f)$" but as "$S(f)$ is a adjoined to $S$" with a topology yet to be described.

We define a topology on $K$ as follows: The set, $\mathscr{B}_S$, of basic open neighborhoods of points in $S$ will be the same as the ones in $S$ when viewed as a topological space on its own. For $x \in S(f)$, $F$ a compact subset of $S$ and $U$ an open neighborhood of $x$ in $T$, we define

$$B_x = (U \cap S(f)) \cup f^{\leftarrow}[U] \backslash F$$

as a basic open neighborhood of $x$. This defines a topology on $K = S \cup^* S(f)$ (where $S(f)$ is adjoined to $S$) generated by the basic open sets,

$$\mathscr{B} = \mathscr{B}_S \cup \{B_x : x \in S(f)\}$$

The set, $K$, will be seen to be a compact set which contains $S$ as a dense subset. These facts will be verified following Notation 22.4 below. In the case where $f$ is not singular (that is, $f[S] \cap T \backslash S(f) \neq \varnothing$), the set $S(f)$ will, however, not be a retract of the compactification

$K = S \cup^* S(f)$.[2] We would like to adopt notation which will help distinguish those compactifications (induced by a function) which are singular from those that are not.

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**Notation 22.4** Suppose $S(f)$ is a singular set of a function $f : S \to T$ where $T$ is compact. If $f$ is not a singular map we will represent the compactification $\gamma S$ induced by $f$ as

$$\gamma S = S \cup^* S(f)$$

Notice how we distinguish it from those compactifications where $f$ is a singular map:

$$\gamma S = S \cup_f S(f)$$

---

We first justify a few statements made in the introductory paragraph above. All properties refer to a continuous function $f : S \to T$ mapping the locally compact Hausdorff space into the compact space $T$.

*We verify that $\gamma S = S \cup^* S(f)$ is a well-defined topological space.* To do this, we will show that the given family, $\mathscr{B}$, of sets satisfies the "base property"(See page 85). If so, it generates a topology on $\gamma S$. Clearly $\gamma S = \cup\{B \in \mathscr{B}\}$. Given

$$A \cap B = (U_x \cup f^{\leftarrow}[U_x] \backslash F_U) \ \cap \ (V_x \cup f^{\leftarrow}[V_x] \backslash F_V)$$

we have,

$$\begin{aligned} x \in A \cap B &= (U_x \cap V_x) \ \cup \ [\,(f^{\leftarrow}[U_x] \backslash F_U) \ \cap \ (f^{\leftarrow}[V_x] \backslash F_V)\,] \\ &= (U_x \cap V_x) \cup [\, f^{\leftarrow}[U_x] \cap f^{\leftarrow}[V_x]\,] \backslash (F_U \ \cup \ F_V) \end{aligned}$$

implies

$$x \in (U_x \cap V_x) \cup [\, f^{\leftarrow}[U_x \cap V_x]\,] \backslash (F_U \ \cup \ F_V) = A \cap B$$

So $\mathscr{B}$ satisfies the base property and so generates a topology on $\gamma S$.

*The set $S$ is dense in $\gamma S = S \cup^* S(f)$.* See that very open neighborhood, $U \cup f^{\leftarrow}[U] \backslash F$, of a point in $S(f)$ intersects $S$.

---

[2] If $f$ is not singular, $f[S]$ may still intersect $S(f)$, or may not even intersect $S(f)$ at all; it is just that $f[S]$ cannot be entirely contained in $S(f)$

*We verify that $\gamma S = S \cup^* S(f)$ is indeed compact and Hausdorff.*

Proof: Given: $\gamma S = S \cup^* S(f)$ and $f : S \to T$ a continuous function mapping completely regular, $S$, into the compact space $T$. Then $f$ extends to $f^{\beta(T)} : \beta S \to T$.

Define the function, $\mu : \beta S \to \gamma S = S \cup^* S(f)$, as:

$$\mu|_{\beta S \setminus S} = f^{\beta(T)}|_{\beta S \setminus S} \quad \text{and} \quad \mu|_S(x) = x$$

We claim that $\mu : \beta S \to S \cup^* S(f)$ is continuous. Let $U \cup f^{\leftarrow}[U] \setminus F$ be an open neighborhood of $y \in S(f) \subseteq \gamma S$ (where $F$ is compact in $S$). Then

$$\begin{aligned} \mu^{\leftarrow}[U \cup f^{\leftarrow}[U] \setminus F] &= \mu^{\leftarrow}[U] \cup \mu^{\leftarrow}[f^{\leftarrow}[U] \setminus F] \\ &= \mu|^{\leftarrow}_{\beta S \setminus S}[U] \cup \mu|^{\leftarrow}_{S}[f^{\leftarrow}[U] \setminus F] \\ &= f^{\beta(T)}|^{\leftarrow}_{\beta S \setminus S}[U] \cup f^{\leftarrow}[U] \setminus F \\ &= f^{\beta(T)}|^{\leftarrow}_{\beta S \setminus S}[U] \cup f^{\beta(T)}|^{\leftarrow}_{S}[U] \setminus F \\ &= f^{\beta(T)\leftarrow}[U] \setminus F \end{aligned}$$

an open subset of $\beta S$. So $\gamma S = S \cup^* S(f)$ is the continuous image of $\beta S$. Then $\gamma S = S \cup^* S(f)$ is compact.

Then $S \cup^* S(f)$ is a compactification of $S$. Also, since $S$ is assumed to be locally compact, $S$ is open in $S \cup^* S(f)$ and is Hausdorff. This is why, we don't normally discuss the compactification $\mathbb{Q} \cup^* S(f)$ induced by a singular function $f$, of the non-locally compact space, $\mathbb{Q}$, of rationals.

*We verify that, if $f$ induces the compactification $\alpha S = S \cup^* S(f)$, then $f : S \to T$ extends continuously to $f^{\alpha} : S \to T$ such that $f^{\alpha}(x) = x$ on $\alpha S \setminus S$.*

Proof: Let $f \in C(S,T)$. Suppose $f^{\alpha}|_S = f$ and $f^{\alpha}(x) = x$ on $\alpha S \setminus S$. Let $x \in S(f) = \alpha S \setminus S$ and suppose $U$ is an open neighborhood of $x$ in $T$. Then $x \in f^{\alpha\leftarrow}[U] = U \cup f^{\leftarrow}[U]$ (by definition of a basic open neighborhood in $\alpha S = S \cup^* S(f)$). Then $f^{\alpha}[U]$ is open in $\alpha S$, so $f^{\alpha}$ is continuous on $\alpha S$, as claimed. *Note that $S(f)$ is not a retract of $\alpha S$ since $f[S] \not\subseteq S(f)$.*

We present another way to view $S(f)$.

**Theorem 22.5** Let $f : S \to T$ be a function mapping a completely regular space $S$ into the compact space $T$. By definition, $S(f) \subseteq \text{cl}_T f[S]$. Then we can describe $S(f)$ as

$$S(f) = f^{\beta(T)}[\beta S \backslash S]$$

*Proof*: Step 1. We first show that $f^{\beta(T)}[\beta S \backslash S] \subseteq S(f)$.

Let $u \in \beta S \backslash S$ and $U$ be an open neighborhood of $f^{\beta(T)}(u)$ in $\text{cl}_T f[S]$. Then $f^{\beta(T)^{\leftarrow}}[U]$ is an open neighborhood of $u$ in $\beta S$. Then $\text{cl}_S f^{\leftarrow}[U]$ cannot be compact (for if $\text{cl}_S f^{\leftarrow}[U]$ is compact and $V$ is an open neighborhood of $u$ in $\beta S$, $(V \cap f^{\beta(T)^{\leftarrow}}[U]) \backslash \text{cl}_S f^{\leftarrow}[U]$ is an open subset of $\beta S$ contained in $\beta S \backslash S$). Then, by definition of $S(f)$, $f^{\beta(T)}(u) \in S(f)$.

Step 2. To establish the claim it suffices to show $S(f) \subseteq f^{\beta(T)}[\beta S \backslash S]$.

Let $x \in S(f)$. To show that $x \in f^{\beta(T)}[\beta S \backslash S]$, it suffices to show $f^{\beta(T)}(y) = x$ for some $y \in \beta S \backslash S$.

Define the function, $\mu : \beta S \to \gamma S = S \cup^* S(f)$, as:

$$\mu|_{\beta S \backslash S} = f^{\beta(T)}|_{\beta S \backslash S} \quad \text{and} \quad \mu|_S(x) = x$$

We showed in #3 above that $\mu$ is continuous. Then

$$\mu^{\leftarrow}(x) = f^{\beta(T)^{\leftarrow}}(x) \cap \beta S \backslash S \subseteq \beta S \backslash S$$

So, for $y \in f^{\beta(T)^{\leftarrow}}(x) \cap \beta S \backslash S$,

$$f^{\beta(T)}(y) = x$$

So

$$x \in f^{\beta(T)}[\beta S \backslash S]$$

Then $S(f) \subseteq f^{\beta(T)}[\beta S \backslash S]$. We are done with Step 2.

Then we conclude,

$$S(f) = f^{\beta(T)}[\beta S \backslash S]$$

as required.

---

*Remark*: We have shown in the proof above that for the given function $f : S \to T$ mapping the completely regular space $S$ into the compact Hausdorff space $T$, the function $f$ extends to $f^{\beta(T)} : \beta S \to T$. Let $K = f^{\beta(T)}[\beta S]$. Then

$$\begin{aligned} f^{\beta(T)}[\beta S] &= f^{\beta(T)}[\beta S \backslash S] \cup f[S] \\ &= S(f) \cup f[S] \\ &= \text{cl}_K f[S] \end{aligned}$$

One should be careful not to confuse the compactification, $S \cup^* S(f)$, of $S$ with the compact set

$$S(f) \cup f[S] = f^{\beta(T)}[\beta S] = \mathrm{cl}_K f[S]$$

It follows from the theorem that $f^{\beta(T)}[\beta S \backslash S] = S(f) \subseteq \mathrm{cl}_K f[S]$.

It is worth noting that, if $f[S]$ turns out to be closed in the compact set $K$, $S(f) = \mathrm{cl}_K f[S] = f[S]$. In this case, since $f[S] \subseteq S(f)$, $f$ is a singular map and so, by definition, $S \cup^* S(f)$ is a singular compactification.

*Remark.* Suppose $g : S \to T$ is a continuous function mapping $S$ into the compact set $T$ . Consider $\alpha S = S \cup^* S(g)$. See that the function $g : S \to T$ extends continuously to

$$g^{\alpha} : S \cup^* S(g) \to cl_T g[S] = g[S] \cup^* S(g)$$

where $g^{\alpha}$ fixes points of $\alpha S \backslash S = S(g)$ (and possibly $S(g) \cap g[S] \neq \varnothing$). Let $\mathscr{K} = C(\mathrm{cl}_T g[S])$ (where $C(\mathrm{cl}_T g[S])$ separates points and closed sets of $\mathrm{cl}_T g[S]$).

Then, if $k \in \mathscr{K}$, $k[g^{\alpha}[\alpha S]] = k[\mathrm{cl}_T g[S]] \subseteq \mathbb{R}$. So,

$$\{k \circ g^{\alpha} : k \in K\} \subseteq C(\alpha S)$$

The function $e_{\mathscr{K}}$ embeds $g^{\alpha}[\alpha S] = \mathrm{cl}_T g[S]$ into $\prod_{k \in \mathscr{K}} [a, b]_k$.

## 22.4 A few examples.

The above definition shows that any continuous function, $f : S \to T$ from $S$ into a compact space, $T$, can be used to construct a compactification of a locally compact Hausdorff space $S$. We consider a few examples to better visualize how this is done.

*Example* 2. Consider the space $\mathbb{R}$ equipped with the usual topology. The space $\mathbb{R}$ is known to be locally compact non-compact Hausdorff. Consider the continuous functions

$$\sin : \mathbb{R} \to [-1, 1] \text{ and } \cos : \mathbb{R} \to [-1, 1]$$

both mapping $\mathbb{R}$ into the compact subspace $[-1, 1]$. Show that sin and cos are both singular maps on $\mathbb{R}$ and so each induce a singular compactification of $\mathbb{R}$.

*Solution*: Case sine: See that, for any $x \in [-1,1]$ and open neighborhood, $U$ of $x$, in $[-1,1]$, $\sin^{\leftarrow}[U]$ is unbounded and so its closure, $\mathrm{cl}_{\mathbb{R}} \sin^{\leftarrow}[U]$, in $\mathbb{R}$ is not compact. Then $x \in S(\sin)$, and so $[-1,1] \subseteq S(\sin)$. By definition, $S(\sin) \subseteq \mathrm{cl}_{[-1,1]} \sin[\mathbb{R}]$, so

$$S(\sin) = \mathrm{cl}_{\mathbb{R}}[\sin[\mathbb{R}]] = [-1,1]$$

Then $\sin : \mathbb{R} \to [-1,1]$ is a singular map. We can then use the sine function to construct the singular compactification

$$\alpha\mathbb{R} = \mathbb{R} \cup_{\sin} S(\sin) = \mathbb{R} \cup_{\sin} [-1,1]$$

of $\mathbb{R}$ with outgrowth $[-1,1]$. The function $\sin : \mathbb{R} \to [-1,1]$ extends continuously to the function

$$\sin^{\alpha} : \alpha\mathbb{R} \to [-1,1]$$

such that $\sin^{\alpha}(x)|_{[-1,1]} = x$.

Case cosine: Proceed similarly to show that $\mathbb{R} \cup_{\cos} S(\cos) = \mathbb{R} \cup_{\cos} [-1,1]$ is also a singular compactification of $\mathbb{R}$.

*Visualizing how points in* $\mathbb{R}$ *converge to points in* $[-1,1] = S(\sin)$. We can visualize what is occurring in these compactifications by examining which sequences converge to which points. For example, for $\frac{\sqrt{2}}{2} \in [-1,1]$, every neighborhood

$$\left(\frac{\sqrt{2}}{2} - \varepsilon, \frac{\sqrt{2}}{2} + \varepsilon\right) \cup \sin^{\leftarrow}\left[\left(\frac{\sqrt{2}}{2} - \varepsilon, \frac{\sqrt{2}}{2} + \varepsilon\right)\right]$$

of $\frac{\sqrt{2}}{2}$ contains the sequence

$$V = \{2n\pi + \frac{\pi}{4} : n = 0, 1, 2, 3, \ldots\}$$

Then $V$ must converge to $\frac{\sqrt{2}}{2}$. Since $\sin^{\alpha} : \alpha\mathbb{R} \to [-1,1]$ is continuous the sequence, $\sin^{\alpha}[V]$, must converge to $\sin^{\alpha}(\frac{\sqrt{2}}{2}) = \frac{\sqrt{2}}{2}$. This is confirmed by that fact that $\sin^{\alpha}[V] = \{\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \ldots, \}$.

The above example produces two compactifications of $\mathbb{R}$ with identical outgrowth. It may be tempting to conclude that they are equivalent compactifications. We will later show (on page 532) that this is not the case.

*Example* 3. We produce an example where $f[S] \cap S(f) = \varnothing$. Consider the space $\mathbb{R}$ equipped with the usual topology. The space $S$ is known to be a locally compact non-compact Hausdorff. Let $T = [-\pi/2, \pi/2]$. Show that $\mathbb{R} \cup^* S(\arctan)$ is *not* a singular compactification of $\mathbb{R}$. Then find the compactification induced by arctan.

*Solution*: We consider the function, $\arctan : \mathbb{R} \to T$. We then verify which points in $\text{cl}_T[\arctan[\mathbb{R}]] = [-\pi/2, \pi/2]$ belong to the singular set $S(\arctan)$. We see that arctan, pulls back open intervals of the form $(a, b)$ in $[-\pi/2, \pi/2]$, to intervals whose closure is compact in $\mathbb{R}$. Then $\arctan[\mathbb{R}] = (-\pi/2, \pi/2) \not\subseteq S(\arctan)$. So arctan is not a singular map. Thus $\mathbb{R} \cup S(\arctan)$ is *not* a singular compactification of $\mathbb{R}$

We now determine $S(\arctan)$. Since the curve of $y = \arctan(x)$ is asymptotic to the horizontal lines $y = -\pi/2$ and $y = \pi/2$, the function arctan pulls backs open intervals of the form $(a, \pi/2]$ and $[-\pi/2, b)$ to unbounded sets and so the "pull backs" of these have non-compact closures in $\mathbb{R}$. So $S(\arctan) = \{-\pi/2, \pi/2\}$. So, even though arctan is not a singular map on $\mathbb{R}$ it does induce a "two-point compactification"

$$\mathbb{R} \cup S(\arctan) = \mathbb{R} \cup \{-\pi/2, \pi/2\}$$

of $\mathbb{R}$. Note that, in this example,

$$f[S] \cap S(\arctan) = (-\pi/2, \pi/2) \cap \{-\pi/2, \pi/2\} = \varnothing$$

*Remark.* We now witnessed two ways of constructing the two-point compactification of $\mathbb{R}$. Recall that, in the last chapter, we used arctan to embed $\mathbb{R}$ into the closed interval $[-\pi/2, \pi/2]$ to form the two-point compactification,

$$\alpha\mathbb{R} = [-\pi/2, \pi/2]$$

of $\mathbb{R}$. In that context, no reference to the singular set $S(\arctan)$ was made. While, in the above example, we constructed the *non-singular* compactification

$$\mathbb{R} \cup^* S(\arctan) = \mathbb{R} \cup \{-\pi/2, \pi/2\}$$

This was done, even though arctan is not a singular map.

*Example* 4. Show that $\beta\mathbb{R}$ is not a singular compactification of $\mathbb{R}$.

*Solution*: We have shown that $\mathbb{R} \cup S(\arctan) = \mathbb{R} \cup \{-\pi/2, \pi/2\}$ is a compactification of $\mathbb{R}$ but not a singular one. Since $\mathbb{R} \cup S(\arctan) \preceq \beta\mathbb{R}$, then, by the example on page 524, $\beta\mathbb{R}$ cannot be singular.

In the following example we show that two compactifications of the same set with the same outgrowth need not be equivalent compactifications.

*Example* 5. Given the two singular compactifications,

$$\begin{aligned} \alpha\mathbb{R} &= \mathbb{R} \cup_{\cos} S(\cos) \\ \gamma\mathbb{R} &= \mathbb{R} \cup_{\sin} S(\sin) \end{aligned}$$

show that, in spite of $S(\sin) = [-1, 1] = S(\cos)$, $\alpha\mathbb{R}$ and $\gamma\mathbb{R}$ are *not* equivalent compactifications.

*Solution*: Suppose $\alpha\mathbb{R}$ and $\gamma\mathbb{R}$ are equivalent compactifications. We will show that this will lead to a contradiction.

Then, by definition of "equivalent compactifications", there exists a homeomorphism,

$$\pi_{\gamma\to\alpha} : \gamma\mathbb{R} \to \alpha\mathbb{R}$$

such that for $x \in \mathbb{R}$, $\pi_{\gamma\to\alpha}(x) = x$.

By definition, the function, $\pi_{\gamma\to\alpha}|_{\gamma\mathbb{R}\backslash\mathbb{R}}$, is a homeomorphism mapping $[-1, 1]$ onto $[-1, 1]$. This means that $\pi_{\gamma\to\alpha}|_{\gamma\mathbb{R}\backslash\mathbb{R}}$ is monotone and maps endpoints to endpoints. Suppose, without loss of generality, that $\pi_{\gamma\to\alpha}$ is increasing and so $\pi_{\gamma\to\alpha}|_{\gamma\mathbb{R}\backslash\mathbb{R}}(-1) = -1$. Then $\pi_{\gamma\to\alpha}|_{\gamma\mathbb{R}\backslash\mathbb{R}}(1) = 1$.

Let

$$\begin{aligned} U &= (a, 1] \subseteq S(\sin) \\ V &= \pi_{\gamma\to\alpha}|_{\gamma\mathbb{R}\backslash\mathbb{R}} U \quad \text{(in } S(\cos)\text{)} \end{aligned}$$

Note that $V$ is a non-empty set. See that we can choose $a$ small enough so that $\sin^{\leftarrow}[U] \cap \cos^{\leftarrow}[V] \cap [-\pi, \pi]$ is empty. Then $\sin^{\leftarrow}[U] \cap \cos^{\leftarrow}[V] = \varnothing$.

Now $\cos : \mathbb{R} \to [-1, 1]$ extends to $\cos^{\alpha} : \alpha\mathbb{R} \to [-1, 1]$ (where $\cos^{\alpha}$ fixes the points of $S(\cos)$).

By definition of open sets in $\gamma\mathbb{R}$, $U \cup \sin^{\leftarrow}[U]$ is open in $\gamma\mathbb{R}$. Then, by continuity of $\cos^{\alpha}$ and $\pi_{\gamma\to\alpha}$ is a homeomorphism, the two sets

$$\begin{aligned} \pi_{\gamma\to\alpha}[U \cup \sin^{\leftarrow}[U]\,] &= V \cup \sin^{\leftarrow}[U] \\ \cos^{\alpha\,\leftarrow}[V] &= V \cup \cos^{\leftarrow}[V] \quad \text{(Since } \cos^{\alpha} \text{ fixes the points of } \alpha\mathbb{R}\backslash\mathbb{R}\text{)} \end{aligned}$$

are both open subsets of $\alpha\mathbb{R}$.

So

$$\begin{aligned}(V \cup \sin^{\leftarrow}[U]) \cap (V \cup \cos^{\leftarrow}[V]) &= V \cup (\sin^{\leftarrow}[U] \cap \cos^{\leftarrow}[V]) \\ &= V \cup \varnothing \\ &= V\end{aligned}$$

a non-empty open subset of $\alpha\mathbb{R}$ which is entirely contained in $\alpha\mathbb{R}\backslash\mathbb{R} = [-1, 1]$. Since $\mathbb{R}$ is dense in $\alpha\mathbb{R} = \mathbb{R} \cup [-1, 1]$, this is impossible. The source of our contradiction is our supposition that $\alpha\mathbb{R}$ and $\gamma\mathbb{R}$ are equivalent. We are done.

*The topologist's since curve.* The real-valued function, $f(x) = \sin(1/x)$, is often referred to, in topological folklore, as the "topologist's sine curve". It is continuous and bounded on its domain $S = \mathbb{R}\backslash\{0\}$. Consider the Stone-Čech compactification $\beta S$. Since $f(x) = \sin(1/x)$ is continuous and bounded on $S$ then it extends to a function $f^{\beta} : \beta S \to \mathbb{R}$ on $\beta S$. It possibly extends to some "smaller" compactification $\alpha S$ of $S$. One is easily mystified when trying to imagine how $f^{\alpha}$ would behave on $\alpha S \backslash S$. The following example sheds a bit of light on this matter.

*Example* 6. Let $S = \mathbb{R}\backslash\{0\}$ be equipped with the standard subspace topology. The function, $f(x) = \sin(1/x)$, is easily seen to be a continuous and bounded function on $S$. Construct and describe a singular compactification $\alpha S$, of $S$ induced by $f(x)$ so that $f$ extends continuously to a function $f^{\alpha} : \alpha S \to \mathbb{R}$.

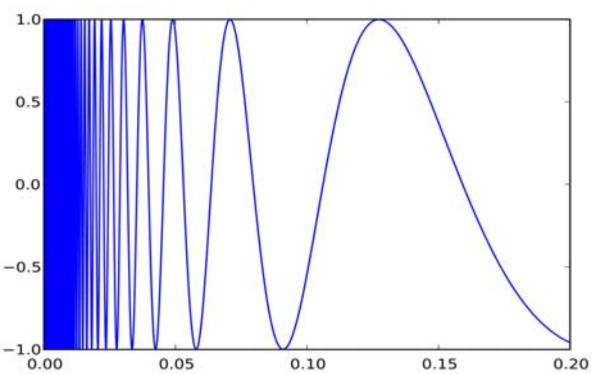


Figure 11: The graph of $y = \sin(1/x)$.

*Solution*: We are given $f(x) = \sin(1/x)$ where $f : S \to T$, maps $S = \mathbb{R}\backslash\{0\}$ onto the compact set $T = [-1, 1]$.

If $y \in [-1,1]$, $U = (y-\varepsilon, y+\varepsilon) \cap [-1,1]$ is an open interval contained in $[-1,1]$. Since $f$ is continuous on $S$ then $f^{\leftarrow}[U]$ is open in $S$. Furthermore, as $x$ approaches zero from both sides, $\mathrm{cl}_S f^{\leftarrow}[U]$ does not attain its limit points near 0 nor at either of the extremities of $\mathbb{R}$. Hence $\mathrm{cl}_S f^{\leftarrow}[U]$ is not compact. Hence $f$ is a singular function on $S$ with $S(f) = [-1,1]$ which induces the singular compactification

$$\alpha S = S \cup_f S(f)$$

The function $f$ then extends to $\alpha S$ where $f^{\alpha}|_{[-1,1]}(x) = x$.

Open neighborhoods of points in $S(f) = [-1,1]$ are of the form $U \cup f^{\leftarrow}[U]$. We witness how sequences in $S$ converge to points in the outgrowth. If $z \in U$, $z = \lim_{n\to\infty} \sin(1/x_n)$ for some sequence $\{x_n\}$ in $S$ converging to zero. For example,

$$\begin{aligned}
\frac{\sqrt{2}}{2} &= \lim_{n\to\infty} \frac{\sqrt{2}}{2} \\
&= \lim_{n\to\infty} \sin(2n\pi + \pi/4) \\
&= \lim_{n\to\infty} \sin\left[1/\left[\frac{1}{(2n\pi+\pi/4)}\right]\right] \\
&= \lim_{n\to\infty} \sin[1/x_n]
\end{aligned}$$

where the terms, $x_n = \frac{1}{(2n\pi+\pi/4)}$, converge to zero as $n$ tends to infinity. Also

$$0 = \lim_{n\to\infty} 0 = \lim_{n\to\infty} \sin\left[1/\left[\frac{1}{(2n\pi)}\right]\right] = \lim_{n\to\infty} \sin[1/x_n]$$

where the terms, $x_n = \frac{1}{2n\pi}$, converge to zero as $n$ tends to infinity.

Also,

$$0 = \lim_{n\to\infty} \sin[1/n] = \lim_{n\to-\infty} \sin[1/n]$$

*Example* 7. We produce a non-singular compactification, $S \cup^* S(f)$, of a space $S$ such that $f[S] \backslash S(f)$ is non-empty. Let $f : \mathbb{R} \to \mathbb{R}$ be the continuous function defined as follows:

$$f(x) = \begin{cases} \sin(x) & \text{if} \quad x \leq 0 \\ \arctan(x) & \text{if} \quad x \geq 0 \end{cases}$$

Then

$$\begin{aligned}
f[\mathbb{R}] &= f[\mathbb{R}^-] \cup f[\mathbb{R}^+] \\
&= \sin[\mathbb{R}^-] \cup \arctan[\mathbb{R}^+] \\
&= [-1,1] \cup [0, \pi/2) \\
&= [-1, \pi/2)
\end{aligned}$$

See that $S(f) = [-1,1] \cup \{\pi/2\}$. Since $f[\mathbb{R}] = [-1,\pi/2) \not\subseteq [-1,1] \cup \{\pi/2\}$ then $f$ is not a singular function and so the corresponding non-singular compactification is $\mathbb{R} \cup^* S(f)$.

Furthermore,

$$\begin{aligned} f[S]\backslash S(f) &= [-1,\pi/2) \setminus [-1,1] \cup \{\pi/2\} \\ &= (1,\pi/2) \\ &\neq \varnothing \end{aligned}$$

22.5 More on equivalent singular compactifications.

We will now produce a characterization of pairs of singular compactifications which are equivalent. But first we must present a few lemmas involving singular sets $S(f)$.

---

**Lemma 22.6** Let $f : S \to K$ be a continuous function mapping a locally compact Hausdorff space into a compact Hausdorff space, $K$, and $Y = \mathrm{cl}_K f[S]$. Then

$$S(f) = \cap\{\mathrm{cl}_Y f[S\backslash F] : F \text{ is compact in } S\}$$

*Proof*: We are given that $f : S \to K$ is a continuous function mapping $S$ into the compact space $K$ and $Y = \mathrm{cl}_K f[S]$. In the proof, $F$ will always represent a compact set in $S$.

Claim #1: We claim that $S(f) \subseteq \mathrm{cl}_Y f[S\backslash F]$ for all compact $F$. To prove this, it suffices to show that $Y\backslash \mathrm{cl}_Y f[S\backslash F] \cap S(f) = \varnothing$.

Let $p \in Y\backslash \mathrm{cl}_Y f[S\backslash F]$. Then there exists an open neighborhood $U$ of $p$ in $Y$ such that $f^{\leftarrow}[U] \subseteq F$. Then $\mathrm{cl}_S f^{\leftarrow}[U]$ is compact so $p \notin S(f)$. Then $Y\backslash \mathrm{cl}_Y f[S\backslash F] \cap S(f) = \varnothing$; so $S(f) \subseteq \mathrm{cl}_Y f[S\backslash F]$, as claimed. We can deduce that

$$S(f) \subseteq \cap\{\mathrm{cl}_Y f[S\backslash F] : F \text{ is compact in } S\}$$

Claim #2: $\cap\{\mathrm{cl}_Y f[S\backslash F] : F \text{ is compact in } S\} \subseteq S(f)$.

Let $p \in \cap\{\mathrm{cl}_Y f[S\backslash F] : F \text{ is compact in } S\}$. Suppose $p \notin S(f)$. Then there is an open neighborhood $U_1$ of $p$ in $Y = \mathrm{cl}_K f[S]$ such that

$\text{cl}_S f^{\leftarrow}[U_1]$ is compact. But

$$\begin{aligned} p &\in \cap\{\text{cl}_Y f[S\setminus F] : F \text{ is compact in } S\} \\ &\subseteq \text{cl}_Y f[S\setminus \text{cl}_S f^{\leftarrow}[U_1]] \\ &\subseteq \text{cl}_Y f[S\setminus f^{\leftarrow}[U_1]] \\ &\subseteq \text{cl}_Y f\circ f^{\leftarrow}[Y\setminus U_1]] \\ &= Y\setminus U \end{aligned}$$

The statement "$p \in Y\setminus U$" contradicts the fact that $U$ is a neighborhood of $p$. Consequently, $\cap\{\text{cl}_Y f[S\setminus F] : F \text{ is compact in } S\} \subseteq S(f)$ as claimed.

So $S(f) = \cap\{\text{cl}_Y f[S\setminus F] : F$ is compact in $S\}$. This completes the proof of the lemma.

---

**Lemma 22.7** Let $S$ be a locally compact Hausdorff space and $\alpha S$ be a compactification of $S$. Let $K$ be a compact space. If $f \in C_\alpha(S,K)$[4], then $f^\alpha[\alpha S\setminus S] = S(f)$.

*Proof*: If $F$ is compact in $S$, then $\alpha S\setminus S \subseteq \text{cl}_{\alpha S}(S\setminus F)$, and so

$$f^\alpha[\alpha S\setminus S] \subseteq f^\alpha[\text{cl}_{\alpha S}(S\setminus F)] \subseteq \text{cl}_{\text{cl}_K f[S]} f[S\setminus F]$$

Then $f^\alpha[\alpha S\setminus S] \subseteq \cap\{\text{cl}_{\text{cl}_K f[S]} f[S\setminus F] : F$ is compact in $S\}$. By the previous lemma $S(f) = \cap\{\text{cl}_{K f[S]} f[S\setminus F] : F$ is compact in $S\}$, so

$$f^\alpha[\alpha S\setminus S] \subseteq S(f)$$

On the other hand, if $p \notin f^\alpha[\alpha S\setminus S]$ and $U$ is an open neighborhood of $p$ in $K$ such that $\text{cl}_K U$ misses $f^\alpha[\alpha S\setminus S]$, then $\text{cl}_S f^{\leftarrow}[U] \subseteq f^{\leftarrow}[\text{cl}_K U]$, a compact subset of $S$. Hence $\text{cl}_S f^{\leftarrow}[U]$ is compact and so $p \notin S(f)$. So

$$S(f) \subseteq f^\alpha[\alpha S\setminus S]$$

We conclude that, if $f$ extends to $f^\alpha : \alpha S \to K$, then $f^\alpha[\alpha S\setminus S] = S(f)$. This proves the lemma.

---

[4]Where $C_\alpha(S,K)$ is the set of continuous functions mapping $S$ into $K$ which extend to $f^\alpha C(\alpha S, K)$.

**Lemma 22.8** Let $f : S \to K$ be a continuous function in $C(S, K)$ mapping the locally compact Hausdorff space $S$ into a compact Hausdorff space, $K$. Suppose $f$ extends to $f^{\alpha(K)} \in C(\alpha S, K)$ for some compactification $\alpha S$ such that $f^{\alpha(K)}$ is one-to-one on $\alpha S \backslash S$. Then

$$\alpha S \equiv S \cup^* S(f)$$

*Proof*: Let $S(f)$ be the singular set of the continuous map $f : S \to K$ (not necessarily a singular function). We are given a compactification, $\alpha S$, such that $f$ extends continuously to $f^{\alpha(K)} : \alpha S \to K$ in such a way that $f^{\alpha}$ is one-to-one on $\alpha S \backslash S$.

By Lemma 22.7, $f^{\alpha(K)}[\alpha S \backslash S] = S(f)$.

If $\gamma S = S \cup^* S(f)$, we define $\pi_{\alpha \to \gamma} : \alpha S \to S \cup^* S(f)$ as follows:

$$\pi_{\alpha \to \gamma}(x) = \begin{cases} f^{\alpha}(x) & \text{if } x \in \alpha S \backslash S \\ x & \text{if } x \in S \end{cases}$$

Claim: That $\pi_{\alpha \to \gamma}$ is continuous on $\alpha S$. It suffices to show that $\pi_{\alpha \to \gamma}$ pulls back open neighborhoods in $S \cup^* S(f)$ to open sets in $\alpha S$.

Suppose $p \in S \cup^* S(f)$. Clearly, if $p \in S$ and $p \in U$ is open in $S$, then $\pi^{\leftarrow}_{\alpha \to \gamma}[U]$ is open in $\alpha S$. If $p \in S(f)$, then an open neighborhood of $p$ is, by definition, of the form $[\, U \cap S(f)\,] \cup f^{\leftarrow}[U] \backslash F$ where $U$ is an open subset of $K$ and $F$ some compact set in $S$. See that,

$$\begin{aligned}
\pi^{\leftarrow}_{\alpha \to \gamma}[\, [\, U \cap S(f)\,] \cup f^{\leftarrow}[U] \backslash F\,] &= \pi^{\leftarrow}_{\alpha \to \gamma}[\, U \cap S(f)\,] \cup \pi^{\leftarrow}_{\alpha \to \gamma}[\, f^{\leftarrow}[U] \backslash F]\,] \\
&= \pi^{\leftarrow}_{\alpha \to \gamma}[U] \cap \pi^{\leftarrow}_{\alpha \to \gamma}[S(f)] \cup f^{\leftarrow}[U] \backslash F \\
&= (f^{\alpha \leftarrow}[U] \cap f^{\alpha \leftarrow}[S(f)]) \cup f^{\leftarrow}[U] \backslash F \\
&= (f^{\alpha \leftarrow}[U \cap S(f)] \cup f^{\leftarrow}[U] \backslash F \\
&= f^{\alpha \leftarrow}[U] \backslash F \\
&= f^{\alpha \leftarrow}[U] \cap S \backslash F
\end{aligned}$$

We see that $\pi_{\alpha \to \gamma}$ pulls back open neighborhoods of points in $S(f)$ to open sets. So $\pi_{\alpha \to \gamma} : \alpha S \to S \cup^* S(f)$ is continuous, as claimed.

By definition, $\alpha S$ and $S \cup^* S(f)$ are equivalent compactifications.

---

We now present an immediate consequence of the above results.

*Example* 8. Suppose $S$ is a completely regular topological space and $\alpha S$ is a compactification of $S$. Then $C(\alpha S)$ separates points and closed sets of $\alpha S$. If $\mathscr{F} = C_\alpha(S)$, the function $e^\alpha_{\mathscr{F}} : \alpha S \to \prod_{f\in\mathscr{F}} \mathbb{R}_f$ is one-to-one on $\alpha S\backslash S$. By the lemma immediately above,

$$e^\alpha_{\mathscr{F}}[\alpha S\backslash S] = S(e_{\mathscr{F}}) \quad \text{and} \quad \alpha S \equiv S \cup^* S(e_{\mathscr{F}})$$

Hence,

*Every compactification $\alpha S$ of $S$ can be expressed in the form*

$$\alpha S \equiv S \cup^* S(e_{C_\alpha}(S))$$

*It is worth emphasizing the following fact of interest.*

Let $\mathscr{F} = C_\alpha(S)$. Suppose $p \in S(e_{\mathscr{F}}) \cap e_{\mathscr{F}}[S]$. Then $p \in e^\alpha_{\mathscr{F}}[\alpha S\backslash S] \cap e_{\mathscr{F}}[S]$. Then $e^{\alpha\leftarrow}_{\mathscr{F}} e^\alpha_{\mathscr{F}}(p)$ is not a singleton. This contradicts the fact that $e^\alpha_{\mathscr{F}}$ is one-to-one on $\alpha S$. So, not only is $e_{\mathscr{F}}$ not a singular map,

$$e_{\mathscr{F}}[S] \cap e^\alpha_{\mathscr{F}}[\alpha S\backslash S] = \varnothing$$

Hence

$$e_{\mathscr{F}}[S] \cap S(e_{\mathscr{F}}) = \varnothing$$

In the particular case where $\mathscr{F} = C^*(S)$

$$\beta S \equiv S \cup^* S(e_{C^*(S)}) \quad \text{and} \quad e_{C^*(S)}[S] \cap e^\beta_{C^*(S)}[\beta S\backslash S] = \varnothing$$

---

**Theorem 22.9** Let $S$ be a completely regular topological space. Suppose $f : S \to K$ and $g : S \to K$ are two continuous *singular* functions mapping $S$ into a compact space $K$. Suppose $S(f)$ and $S(g)$ are homeomorphic. Then the two induced singular compactifications,

$$\begin{aligned} \alpha S &= S \cup_f S(f) \\ \gamma S &= S \cup_g S(g) \end{aligned}$$

are equivalent if and only if the singular function $f : S \to S(f)$ of $\alpha S$ extends continuously to $f^\gamma : \gamma S \to S(g)$ such that $f^\gamma$ separates the points of $\gamma S\backslash S = S(g)$.

*Proof*: We are given that $f : S \to K$ and $g : S \to K$ are two continuous *singular* functions mapping $S$ into a compact space $K$ inducing the two singular compactifications, $\alpha S = S \cup_f S(f)$ and $\gamma S = S \cup_g S(g)$. Also, $S(f)$ and $S(g)$ are seen to be homeomorphic.

( $\Rightarrow$ ) Suppose $\alpha S = S \cup_f S(f)$ and $\gamma S = S \cup_g S(g)$ are equivalent.

We are required to show that the singular function $f : S \to S(f)$ extends continuously to $f^\gamma : \gamma S \to S(g)$ such that $f^\gamma$ separates the points of $\gamma S \backslash S = S(g)$.

Recall that $f^\alpha : \alpha S \to \alpha S \backslash S$ acts as the identity map on $\alpha S \backslash S$. Also, since $\alpha S$ and $\gamma S$ are equivalent, then there is a continuous map $\pi_{\gamma\to\alpha} : \gamma S \to \alpha S$ such that $\pi_{\gamma\to\alpha}(x) = x$ on $S$ and $\pi_{\gamma\to\alpha}$ maps $\gamma S\backslash S$ homeomorphically onto $\alpha S\backslash S$.

Let $f^\gamma : \gamma S \to S(f)$ be defined as follows:

$$f^\gamma = f^\alpha \circ \pi_{\gamma\to\alpha}$$

Then $f^\gamma$ is continuous, $f^\gamma|_S = f$ and $f^\gamma|_{S(g)} = \pi_{\gamma\to\alpha}|_{S(g)}$. This shows that $f : S \to S(f)$ extends continuously to the function $f^\gamma : S \cup_g S(g) \to S(f)$. Since $\pi_{\gamma\to\alpha}$ is a homeomorphism on $S(g)$ and $f^\alpha$ is the identity function on $S(f)$, then $f^\gamma$ separates points of $S(g)$, as required.

( $\Leftarrow$ ) We are given that both $f$ and $g$ are singular maps on $S$ and $\alpha S = S \cup_f S(f)$ and $\gamma S = S \cup_g S(g)$. We are also given that the singular function $f : S \to S(f)$ extends continuously to $f^\gamma : S \cup_g S(g) \to S(g)$ such that $f^\gamma$ separates the points of $S(g)$.

Then, by Lemma 22.8, $S \cup_g S(g)$ is a compactification which is equivalent to $S \cup^* S(f)$.

Since $f$ is singular, $S \cup^* S(f) = S \cup_f S(f)$. So $S \cup_f S(f)$ and $S \cup_g S(g)$ are equivalent.

---

**Theorem 22.10** Two continuous functions, $f : S \to K$ and $g : S \to K$, will be said to be *homeomorphically related* if there exists a homeomorphic function $h : cl_K g[S] \to cl_K h[S]$ such that $h(g(x)) = f(x)$ for all $x \in S$. Suppose that $f : S \to K$ and $g : S \to K$ are two singular maps such that $S(f) = S(g)$. If $f$ and $g$ are homeomorphically related, then $S \cup_f S(f)$ and $S \cup_g S(g)$ are equivalent compactifications.

*Proof*: We are given two continuous functions, $f : S \to K$ and $g : S \to K$, mapping $S$ into the compact space $K$.

Suppose $f$ and $g$ are homeomorphically related singular maps which, respectively, induce the singular compactifications,

$$\begin{aligned} \alpha S &= S \cup_f S(f) \\ \gamma S &= S \cup_g S(g) \end{aligned}$$

where $S(f) = S(g)$. By definition, there exists a homeomorphism $h : S(g) \to S(f)$ such that $h(g(x)) = f(x)$.

See that $g : S \to S(g)$ extends continuously to $g^\gamma : S \cup_g S(g) \to S(g)$ where $g^\gamma$ is the identity map when restricted to $S(g)$. Then $h \circ g : S \to S(f)$ extends to

$$(h \circ g)^\gamma : S \cup_g S(g) \to S(f)$$

where $(h \circ g^\gamma)|_{S(g)}(x) = h(x)$. Since $(h \circ g)(x) = f(x)$ on $S$, $f$ extends to $(h \circ g)^\gamma = f^\gamma$ where

$$f^\gamma : S \cup_g S(g) \to S(f)$$

and $f^\gamma = h$ on $S(g)$. So $f^\gamma$ separates points of $S(g)$.

By Theorem 22.9, $S \cup_f S(f)$ and $S \cup_g S(g)$ are equivalent, as required.

---

We can now present an example where this particular concept plays a key role.

*Example* 9. The two compactifications $\mathbb{R} \cup_{\sin^2} S(\sin^2)$ and $\mathbb{R} \cup_{\cos^2} S(\cos^2)$ are easily seen to be singular compactifications with outgrowth $[0, 1]$. Show that they are equivalent compactifications.

*Solution*: Consider the function $h(x) = 1 - x$ mapping $[0, 1]$ onto $[0, 1]$. It is a one-to-one continuous function. Also, note that

$$h(\sin^2(x)) = 1 - \sin^2(x) = (1 - \sin^2)(x) = \cos^2(x)$$

on $[0, 1]$. Then $\sin^2$ and $\cos^2$ are homeomorphically related. By the above theorem $\mathbb{R} \cup_{\sin^2} S(\sin^2)$ and $\mathbb{R} \cup_{\cos^2} S(\cos^2)$ are equivalent compactifications. This is what we were required to show.

There can be various ways of showing that ...

> *... the compactification $\beta\mathbb{N}$ is not a singular compactification.*

In the next example we propose one method.

*Example* 10. Show that $\beta\mathbb{N}$ cannot be a singular compactification.

*Solution*: Suppose $\beta\mathbb{N}$ is a singular compactification, Then there is a retraction function $r : \beta\mathbb{N} \to \beta\mathbb{N}\backslash\mathbb{N}$ which maps $\beta\mathbb{N}$ onto $\beta\mathbb{N}\backslash\mathbb{N}$. We know that $\beta\mathbb{N}$ is separable, but on page 508, we showed that $\beta\mathbb{N}\backslash\mathbb{N}$

is not a separable space. By Theorem 6.11 we know that continuous images of separable spaces are separable. So $\beta\mathbb{N}\backslash\mathbb{N}$ cannot be the continuous image of $\beta\mathbb{N}$. So $\beta\mathbb{N}$ is not a singular compactification.

---

**Theorem 22.11** The Stone-Čech compactification $\beta\mathbb{R}$ contains an uncountable family of pairwise disjoint copies of $\beta\mathbb{N}$ whose union is dense in $\beta\mathbb{R}$.

*Proof* : We have seen that

$$\alpha\mathbb{R} = \mathbb{R} \cup_{\sin} S(\sin) = \mathbb{R} \cup_{\sin} [-1, 1]$$

is a singular compactification of $\mathbb{R}$ generated by the singular function, $\sin : \mathbb{R} \to [-1, 1]$ (see the example on page 532). Then $\alpha\mathbb{R}\backslash\mathbb{R}$ is a retract of $\alpha\mathbb{R}$, where $\sin^{\alpha}|_{\alpha\mathbb{R}\backslash\mathbb{R}}(x) = x$ on the outgrowth $[-1, 1]$.

Note that, for each $x \in [-1, 1]$, the set $\sin^{\leftarrow}(x)$ is a countably infinite discrete closed subspace of $\mathbb{R}$ (since it contains none of its limit points). Since $\sin^{\leftarrow}(x)$ is unbounded in $\mathbb{R}$, $\sin^{\leftarrow}(x)$ is not compact. Then, for each $x \in [-1, 1]$, the subspace, $\sin^{\leftarrow}(x)$, can be viewed as a homeomorphic copy of $\mathbb{N}$ contained in $\mathbb{R}$.

See that $\mathrm{cl}_{\alpha\mathbb{R}} \sin^{\leftarrow}(x)$ intersects $\alpha\mathbb{R}\backslash\mathbb{R}$.

If $a \in [-1, 1]$, $\sin^{\alpha\leftarrow}(a) = \{a\} \cup \sin^{\leftarrow}(a)$ is compact in $\alpha\mathbb{R}$ and so contains $\mathrm{cl}_{\alpha\mathbb{R}} \sin^{\leftarrow}(a)$. Then

$$\mathrm{cl}_{\alpha\mathbb{R}} \sin^{\leftarrow}(a) = \{a\} \cup \sin^{\leftarrow}(a)$$

Since $\mathbb{R}$ is metrizable, every $\mathbb{R}$-closed subset of the form $\sin^{\leftarrow}(a)$ is $C^*$-embedded (see the example on page 485). Then, by Theorem 21.21,

$$\mathrm{cl}_{\beta\mathbb{R}} \sin^{\leftarrow}(a) = \beta[\sin^{\leftarrow}(a)]$$

See that,

$$\begin{aligned} \pi_{\beta\to\alpha}[\beta[\sin^{\leftarrow}(a)]] &= \pi_{\beta\to\alpha}[\mathrm{cl}_{\beta\mathbb{R}} \sin^{\leftarrow}(a)] \\ &= \mathrm{cl}_{\alpha\mathbb{R}}\pi_{\beta\to\alpha}[\sin^{\leftarrow}(a)] \\ &= \mathrm{cl}_{\alpha\mathbb{R}} \sin^{\leftarrow}(a) \\ &= \{a\} \cup \sin^{\leftarrow}(a) \end{aligned}$$

So for each $x \in [-1, 1]$,

$$\beta[\sin^{\leftarrow}(x)] \subseteq \pi^{\leftarrow}_{\beta\to\alpha}[\{x\} \cup \sin^{\leftarrow}(x)]$$

Since $\{\pi^{\leftarrow}_{\beta\to\alpha}[\{x\} \cup \sin^{\leftarrow}(x)] : x \in [-1,1]\}$ partitions $\beta\mathbb{R}$, then the elements of the family

$$\mathscr{F} = \{\beta[\sin^{\leftarrow}(x)] : x \in [-1,1]\}$$

are pairwise disjoint.

Since,

$$\begin{aligned}\mathbb{R} &\subseteq \cup\{\sin^{\leftarrow}(x) : x \in [-1,1]\}\\ &\subseteq \cup\{\mathrm{cl}_{\beta\mathbb{R}} \sin^{\leftarrow}(x) : x \in [-1,1]\}\end{aligned}$$

then $\mathbb{R} \subseteq \cup\mathscr{F}$. Hence $\cup\mathscr{F}$ is dense in $\beta\mathbb{R}$.

We conclude that $\beta\mathbb{R}$ contains the union of an uncountable family of pairwise disjoint copies of $\beta\mathbb{N}$ which is dense in $\beta\mathbb{R}$, as required.

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**Corollary 22.12** The subset $\beta\mathbb{R}\backslash\mathbb{R}$ of $\beta\mathbb{R}$ contains an uncountable family of pairwise disjoint copies of $\beta\mathbb{N}$.

*Proof*: Let $a \in [-1,1]$. We have shown that since the function $\sin : \mathbb{R} \to [-1,1]$ is singular it induces the singular compactification $\alpha\mathbb{R} = \mathbb{R} \cup [-1,1]$. We then showed that $\mathrm{cl}_{\beta\mathbb{R}} \sin^{\leftarrow}(a)$ is equivalent to the compactification $\beta[\sin^{\leftarrow}(a)]$.

By Theorem 21.20, the cardinality of $\beta[\sin^{\leftarrow}(a)]\backslash\sin^{\leftarrow}(a)$ is $2^c$.

Let $x_1$ and $x_2$ be distinct points in $\beta[\sin^{\leftarrow}(a)] \setminus \sin^{\leftarrow}(a)$. Then there exists disjoint open neighborhoods $B_1$ and $B_2$ which separate $x_1$ and $x_2$ (since $\beta[\sin^{\leftarrow}(a)] \setminus \sin^{\leftarrow}(a)$ is compact Hausdorff).

Suppose $D_n = \{x_i : i = 1 \text{ to } n\}$ are $n$ distinct points and $x_{n+1}$ belongs to $\beta[\sin^{\leftarrow}(a)]\backslash\sin^{\leftarrow}(a)$ but not in $D_n$. Since $\beta[\sin^{\leftarrow}(a)]\backslash\sin^{\leftarrow}(a)$ is completely regular, there exist disjoint open neighborhoods of $\{B_i : i = 1 \text{ to } n+1\}$ which separate the points of $D_{n+1} = \{x_i : i = 1 \text{ to } n+1\}$. We can then inductively construct the set $D = \{x_i : i = 12, 3, \ldots\}$ for which we have a set $\{B_i : i = 1, 2, 3, \ldots\}$ of pairwise disjoint open neighborhoods in $\beta[\sin^{\leftarrow}(a)]\backslash\sin^{\leftarrow}(a)$ which separate the points of $D$.

We have shown that $\beta[\sin^{\leftarrow}(a)]\backslash\sin^{\leftarrow}(a)$ contains a countably infinite discrete space $D$. We claim that

$$\mathrm{cl}_{\beta[\sin^{\leftarrow}(a)]} D = \beta D$$

in $\beta[\sin^{\leftarrow}(a)]\backslash\sin^{\leftarrow}(a)$. If so, then $\beta[\sin^{\leftarrow}(a)]\backslash\sin^{\leftarrow}(a)$ contains a copy of $\beta\mathbb{N}$.

Proof of claim: To prove this, it suffices to show that $D$ is $C^*$-embedded in $\beta[\sin^{\leftarrow}(a)]$. Let $H = D \cup \sin^{\leftarrow}(a)$, a subset of $\beta[\sin^{\leftarrow}(a)]$. The set $H$ is regular and is easily seen to be Lindelöf; then $H$ is is normal (see Theorem 16.6). Since $\sin^{\leftarrow}(a)$ is discrete in $H$ none of its points are limit points of $D$. So $D$ is closed in $H$. From Theorem 10.9 we obtain that $D$ is $C^*$-embedded in $H$.

Suppose $f \in C^*(D)$. Then $f$ extends to $f^* \in C^*(H)$. Then $f^*|_{\sin^{\leftarrow}(a)}$ is continuous on $\sin^{\leftarrow}(a)$. Since $\sin^{\leftarrow}(a)$ is $C^*$-embedded in $\beta[\sin^{\leftarrow}(a)]$, then $f^*|_{\sin^{\leftarrow}(a)}$ extends to a function $g \in C(\beta[\sin(a)])$. So $f \in C^*(D)$ extends to $g \in \beta[\sin(a)]$.

We then can conclude that $\mathrm{cl}_{\beta[\sin^{\leftarrow}(a)]} D = \beta D$ is a subset of the set $\beta[\sin^{\leftarrow}(a)] \setminus \sin^{\leftarrow}(a)$, as claimed.

Then, for each $x \in [-1, 1]$, $\beta[\sin^{\leftarrow}(x)] \setminus \sin^{\leftarrow}(x)$ contains a copy of $\beta\mathbb{N}$.

So $\beta\mathbb{R}\backslash\mathbb{R}$ contains an uncountable family of pairwise disjoint copies of $\beta\mathbb{N}$.

---

In the following example we see how we can use a singular function to construct, from a rectangle, a cylindrical shell.

*Example* 11. Consider the non-compact subspace, $S = [0, 2\pi] \times (0, 2\pi)$ of $\mathbb{R}^2$ equipped with the usual topology (a rectangle with the top and bottom edges removed). Consider the real-valued function $f : S \to [0, 2\pi]$ defined as $f[\{x\} \times (0, 2\pi)] = \{x\}$ for each $x \in [0, 2\pi]$. For example, $f(0, y) = 0$, $f(\pi, y) = \pi$, $f(1, y) = 1$, for each $y \in (0, 2\pi)$. Verify that the function $f$ is a singular function, hence $f$ induces a singular compactification. Also, show that the singular compactification of $S$, induced by $f$, can be described (topologically speaking) as a closed and bounded cylindrical shell with radius 1 and height $2\pi$.

*Solution*: We are given that $f : S \to [0, 2\pi]$ is defined as $f(x, y) = x$. For $u \in [0, 2\pi]$ and $B_\varepsilon(u) = \{x \in [0, 2\pi] : |x - u| < \varepsilon\}$ and $f^{\leftarrow}[u] = \{u\} \times (0, 2\pi)$. Since $f^{\leftarrow}[B_\varepsilon(u)] = B_\varepsilon(u) \times (0, 2\pi)$, $f$ is open in $S$ and so $f$ is continuous on $S$. See that, for all $x \in [0, 2\pi]$,

$$\mathrm{cl}_S f^{\leftarrow}[B_\varepsilon(x)] = \mathrm{cl}_{\mathbb{R}} B_\varepsilon(x) \times (0, 2\pi)$$

is not compact in $S$ so, by definition, $S(f) = \mathrm{cl}_{\mathbb{R}} f[S] = [0, 2\pi]$. Since $f[S]$ is a subset of $S(f) = [0, 2\pi]$, then $f$ is a singular map and so $S \cup_f [0, 2\pi]$ is a singular compactification of $S$.

We now describe the compactification $S \cup_f [0, 2\pi]$. Let's consider the point $x$ in $S(f)$ viewed as an element of the compactification $S \cup_f S(f)$. Let $F_\delta$ represent the subset of $S$ of the form

$$F_\delta = [0, 2\pi] \times [\delta, 2\pi - \delta]$$

See that $F_\delta$ is a compact subset of $S$. If $x \in S(f) = [0, 2\pi]$ an open neighborhood base element of $x$ in $S \cup_f S(f)$ would look something like this

$$B_\varepsilon(x) \ \cup \ \ f^{\leftarrow}[B_\varepsilon(x)] \setminus F_\delta$$

where $f^{\leftarrow}[B_\varepsilon(u)] = (u - \varepsilon, u + \varepsilon) \times (0, 2\pi)$.

Let $u \in S(f) = [0, 2\pi]$. Then,

$$\mathscr{B}_u = \left\{B_{1/n}(u) \ \cup \ f^{\leftarrow}[B_{1/n}(u)] \setminus [0, 2\pi] \times \left[\tfrac{1}{n}, 2\pi - \tfrac{1}{n}\right] : n = 1, 2, 3, \ldots\right\}$$

forms an open neighborhood base of $u$ in $S \cup_f S(f)$.
See that

$$\cap_{n \in \mathbb{N} \setminus \{0\}} \mathscr{B}_u = \{u\}$$

So $S(f) = [0, 2\pi]$ appears as the "edge" which provides the material necessary to seal together the bottom and the top edges of the rectangle $[0, 2\pi] \times (0, 2\pi)$ to form a cylindrical shell.

*Example* 12. Consider the non-compact subspace, $S = [0, 2\pi] \times (0, 2\pi)$ of $\mathbb{R}^2$ equipped with the usual topology. Consider the function $g : S \to [0, 2\pi]$ defined as

$$g[\{x\} \times (0, 2\pi)] = \{2\pi - x\}$$

Verify that that $S \cup_g S(g)$ (where $S(g) = [0, 2\pi]$) is a singular compactification.
Also, if $f : S \to [0, 2\pi]$ is the function as defined in the Example 9 above, show that $S \cup_f S(f)$ and $S \cup_g S(g)$ are not equivalent compactifications (in spite of the fact that $S(f) = S(g) = [0, 2\pi]$).

*Solution*: We are given that $S = [0, 2\pi] \times (0, 2\pi)$ and the function $g : S \to [0, 2\pi]$ is defined as $g(x, y) = 2\pi - x$. Also, $g[S] = [0, 2\pi]$. Since $g^{\leftarrow}[[0, 2\pi]] = [0, 2\pi] \times (0, 2\pi)$ is not compact then $g$ is a singular map and induces a singular compactification $S \cup_g [0, 2\pi]$. If $u \in [0, 2\pi]$, $g^{\leftarrow}[u] = \{2\pi - u\} \times (0, 2\pi)$.

For $u \in S(g) = [0, 2\pi]$, an open neighborhood base element of $u$ in $S \cup_g S(g)$ would look something like this

$$(u - \varepsilon, u + \varepsilon) \ \cup \ \ g^{\leftarrow}[(u - \varepsilon, u + \varepsilon)] \setminus F_\delta$$

where

$$F_\delta = [0, 2\pi] \times [\delta, 2\pi - \delta]$$

and $g^{\leftarrow}[(u-\varepsilon, u+\varepsilon)] = (2\pi - (u-\varepsilon),\, 2\pi - (u+\varepsilon)) \times (0, 2\pi)$.

Let $u \in S(g) = [0, 2\pi]$. Then,

$$\mathscr{B}_u = \left\{B_{1/n}(u) \ \cup \ g^{\leftarrow}[B_{1/n}(u)] \backslash [0, 2\pi] \times \left[\tfrac{1}{n},\, 2\pi - \tfrac{1}{n}\right] : n = 1, 2, 3, \ldots\right\}$$

forms an open neighborhood base of $u$ in $S \cup_g S(g)$.

See that

$$\cap_{n \in \mathbb{N} \backslash \{0\}} \mathscr{B}_u = \{u\}$$

Let $\gamma S = S \cup_f S(f)$ and $\alpha S = S \cup_g S(g)$.

We claim that: The compactifications, $\gamma S$ and $\alpha S$ are not equivalent compactifications.

Proof of claim: Suppose $\gamma S \equiv \alpha S$. Then there exists a homeomorphism $\pi_{\gamma \to \alpha} : \gamma S \to \alpha S$ which fixes the points of $S$. Then $\pi_{\gamma \to \alpha}|_{\gamma S \backslash S}$ maps $[0, 2\pi]$ homeomorphically onto $[0, 2\pi]$ and so is monotone. Suppose without loss of generality that it is increasing and so maps 0 to 0 and $2\pi$ to $2\pi$.

Consider the open ball $B = B_\varepsilon(2\pi) = (2\pi - \varepsilon, 2\pi]$ in $[0, 2\pi]$. Let

$$D = \pi_{\gamma \to \alpha}[B] = (2\pi - \varepsilon, 2\pi]$$

an open subset of $S(g)$. Then

$$\begin{aligned} f^{\leftarrow}[B] &= (2\pi - \varepsilon, 2\pi] \times (0, 2\pi) \\ g^{\leftarrow}[D] &= [0, \varepsilon) \times (0, 2\pi) \end{aligned}$$

We can of course choose $\varepsilon$ so that $f^{\leftarrow}[B] \cap g^{\leftarrow}[D] = \varnothing$.

Recall that $g : S \to S(g)$ extends to $g^\alpha : \alpha S \to S(g)$ such that $g^\alpha$ fixes the points of $S(g)$, so we have

$$\begin{aligned} g^{\alpha\leftarrow}[D] &= D \cup g^{\leftarrow}[D] \quad \text{(An open subset of } \alpha S) \\ \pi_{\gamma \to \alpha}[B \cup f^{\leftarrow}[B]] &= D \cup f^{\leftarrow}[B] \quad \text{(An open subset of } \alpha S) \end{aligned}$$

Then

$$\begin{aligned} (D \cup g^{\leftarrow}[D]) \cap (D \cup f^{\leftarrow}[B]) &= D \cup (g^{\leftarrow}[D]) \cap f^{\leftarrow}[B]) \\ &= D \cup \varnothing \\ &= D \end{aligned}$$

so $D$ is an open subset of $\alpha S$ which is contained in $\alpha S \backslash S$. A contradiction! So $\gamma S \not\equiv \alpha S$. [6]

*Example* 13 Consider the non-compact subspace,

$$S = \{(\cos(x), \sin(x), z) : x \in [0, 2\pi], z \in (0, 1)\}$$

in $\mathbb{R}^3$ equipped with the usual topology. The set $S$ is a cylinder of radius 1 and height 1 (with the top and bottom removed). Consider the function $f : S \to \mathbb{R}^2$ defined as

$$f(\cos(x), \sin(x), z) = (\cos(x), \sin(x))$$

for $x \in [0, 2\pi]$. Verify that the function $f$ is a singular function and determine its singular set, $S(f)$. Describe the singular compactification, $S \cup_f S(f)$.

*Solution*: Let

$$D = \{(\cos(x), \sin(x)) : x \in [0, 2\pi]\}$$

We are given that $f$ maps $S$ onto $D$. For $\varepsilon > 0$ and $(u, v) \in D$, let $B_\varepsilon(u, v) = \{(a, b) \in D : \|(u - a, v - b)\|_2 < \varepsilon\}$. Hence the set

$$f^{\leftarrow}[B_\varepsilon(u, v)] = \{(a, b, z) \in S : \|(u - a, v - b)\|_2 < \varepsilon\}$$

is open in $S$. So $f$ is continuous on $S$.

We claim that $f$ is a singular map and so induces the singular compactification, $S \cup_f D$.

Proof of claim: Let $\overline{B_\varepsilon(u, v)} = \{(a, b) \in D : \|(u - a, v - b)\|_2 \leq \varepsilon\}$. For any $(u, v) \in D$ see that,

$$\mathrm{cl}_S f^{\leftarrow}[B_\varepsilon(u, v)] = \overline{B_\varepsilon(u, v)} \times (0, 1)$$

is not compact in $S$. Then, by definition, $S(f) = D$. Since $f[S] = D$ then $f$ is a singular map and so $S \cup_f D$ is a singular compactification of $S$. This establishes the claim.

We now describe $S \cup_f S(f)$.
Let $(u, v) \in S(f) = D$. Then the family of $S \cup_f S(f)$ subsets,

$$\mathscr{B}_{(u,v)} = \left\{B_{1/n}(u, v) \ \cup \ f^{\leftarrow}[B_{1/n}(u, v)] \backslash \left[D \times [\tfrac{1}{n}, 1 - \tfrac{1}{n}]\,\right] : n = 1, 2, 3, \ldots\right\}$$

[6] See that the compactification $S \cup_g S(g)$ constructed in this way is a Möbius strip.

forms an open neighborhood base of $(u, v)$ in $S \cup_f S(f)$.
See that

$$\cap_{n\in\mathbb{N}\backslash\{0\}}\mathscr{B}_{(u,v)} = \{(u, v)\}$$

So $S(f) = D$ appears as the "circle" which provides the material necessary to seal together the bottom and the top edges of the cylindrical shell of radius 1 and height 1, $D \times (0, 1)$, to form a "torus" (topologically speaking).

*Example* 14 Let $S = \mathbb{R}^2$ be equipped with the usual topology. Let $p$ be a point not in $S$ and $f : S \to \{p\}$ be the function defined as $f[S] = \{p\}$. Describe the space $S \cup^* S(f)$. Also, describe a resulting open base for the point $p$ in the compactification $S \cup \{p\}$. Provide a geometric interpretation of $S \cup \{p\}$.

*Solution*: We first note that $S$ is non-compact, locally compact Hausdorff, and that the function $f : S \to \{p\}$ is a continuous singular map, with $S(f) = \{p\}$ (since $\text{cl}_S f^{\leftarrow}\{p\} = \text{cl}_S S = S$ a non-compact set).

We describe an open neighborhood base for the point in the outgrowth of $S \cup \{p\}$. For each $\delta > 0$, let $F_\delta = \{(x, y) \in S : \|(x, y)\|_2 \leq \delta\}$. The set

$$\mathscr{B}_p = \{ \ \{p\} \cup S\backslash F_\delta : \delta > 0 \ \}$$

then represents a family of open neighborhoods of $p$ in $S \cup_f S(f)$. Then, $\cap_{\delta>0}\mathscr{B}_p = \{p\}$. So $S \cup_f S(f) = \mathbb{R}^2 \cup_f \{p\}$ is a one-point compactification of $\mathbb{R}^2$.

The 2-sphere,

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : \|(x, y, z)\| = 1\}$$

discussed in Theorem 21.16 was also viewed as the one-point compactification of $\mathbb{R}^2$.
Since any two one-point compactifications were shown to be equivalent, $\mathbb{R}^2 \cup_f \{p\}$ and the 2-sphere, $S^2$, are homeomorphic.

---

**Theorem 22.13** Suppose $S \cup_g S(g)$ is a singular compactification. Let $K = S(g)$ and $\mathscr{K} = C(K)$. Then $S \cup_{e_{\mathscr{K}}\circ g} S(e_{\mathscr{K}}\circ g)$ is a singular compactification. Furthermore, $S \cup_{e_{\mathscr{K}}\circ g} S(e_{\mathscr{K}}\circ g)$ and $S \cup_g S(g)$ are equivalent compactifications.

*Proof*: We are given that $S \cup_g S(g)$ is a singular compactification and that $K = S(g)$. Suppose $\mathscr{K} = C(K)$.

We are required to show that $S \cup_{e_{\mathscr{K}} \circ g} S(e_{\mathscr{K}} \circ g)$ and $S \cup_g S(g)$ are equivalent compactifications.

Since $K = S(g)$ is compact Hausdorff it is completely regular. If $\mathscr{K} = C(K)$, the evaluation map

$$e_{\mathscr{K}} : S(g) \to T = \prod_{k \in \mathscr{K}} [a_k, b_k]$$

embeds $S(g)$ into the product space $T = \prod_{k \in \mathscr{K}} [a_k, b_k]$ where

$$\begin{aligned} e_{\mathscr{K}}[g[S]] &\subseteq e_{\mathscr{K}}[S(g)] \\ &= < k\,[S(g)] >_{k \in \mathscr{K}} \\ &\subseteq \prod_{k \in \mathscr{K}} [a_k, b_k] \end{aligned}$$

We also have,

$$\begin{aligned} e_{\mathscr{K}}[g^{\alpha}[\alpha S]] &= < k[g^{\alpha}[\mathrm{cl}_{\alpha S} S]] >_{k \in \mathscr{K}} \\ &= < k[\mathrm{cl}_K g[S] >_{k \in \mathscr{K}} \\ &= < \mathrm{cl}_{\mathbb{R}} k[g[S]] >_{k \in \mathscr{K}} \\ &= \mathrm{cl}_T[e_{\mathscr{K}}[g[S]]] \\ &\subseteq T = \prod_{k \in \mathscr{K}} [a_k, b_k] \end{aligned}$$

So $e_{\mathscr{K}}[g^{\alpha}[\alpha S]] = \mathrm{cl}_T[e_{\mathscr{K}}[g[S]]]$.

So $e_{\mathscr{K}}[g[S]]$ is dense in $e_{\mathscr{K}}[g^{\alpha}[\alpha S]]$.

We claim that

$$S \cup_{e_{\mathscr{K}} \circ g} S(e_{\mathscr{K}} \circ g)$$

is a singular compactification of $S$. It suffices to show that $\mathrm{cl}_T[e_{\mathscr{K}}[g[S]]]$ is a singular set of $e_{\mathscr{K}} \circ g$. Suppose $x \in \mathrm{cl}_T[e_{\mathscr{K}}[g[S]]]$ and $U$ is an open neighborhood of $x$ in $\mathrm{cl}_T[e_{\mathscr{K}}[g[S]]]$. Since $e_{\mathscr{K}}$ maps $S(g)$ homeomorphically into $\mathrm{cl}_T[e_{\mathscr{K}}[g[S]]]$, then $e_{\mathscr{K}}{}^{\leftarrow}[U]$ is open in $S(g)$, hence $g^{\leftarrow}[e_{\mathscr{K}}{}^{\leftarrow}[U]]$ has non-compact closure in $S$. Then $(e_{\mathscr{K}} \circ g)^{\leftarrow}[U]$ has non-compact closure in $S$. Then $\mathrm{cl}_T[e_{\mathscr{K}}[g[S]]] = S(e_{\mathscr{K}} \circ g)$. We conclude that $S \cup_{e_{\mathscr{K}} \circ g} S(e_{\mathscr{K}} \circ g)$ is a singular compactification.

This follows immediately from Theorem 22.10 that

$$S \cup_{e_{\mathscr{K}} \circ g} S(e_{\mathscr{K}} \circ g) \equiv S \cup_g S(g)$$

22.6 What kind of space has only singular compactifications?

We have already shown a compactification "less than" a singular compactification must be singular. So, if $\beta S$ is a singular compactification of $S$, then all compactifications of $S$ are singular. We wonder what class of topological spaces satisfies this property?

---

**Theorem 22.14** Let $S$ be locally compact, non-compact and Hausdorff. If $\beta S$ is a singular compactification, then $S$ is pseudocompact.[7]

*Proof*: The proof of the statement is deferred to the end of the next section in Theorem 23.9.

---

We know of one topological space $S$ for which $\beta S$ is a singular compactification. We have shown that if $S = [0, \omega_1)$ then $\beta S = [0, \omega_1] = \omega S$. Inspired by this particular example we exhibit a topological space $S$ whose, $\beta S$, is singular but not equivalent to $\omega S$.

---

**Theorem 22.15** Let $T$ be a compact Hausdorff space of cardinality $\kappa$ and suppose $\omega_n$ is a limit ordinal (where $n \in \mathbb{N}\backslash\{0\}$) such that $\omega_n > \kappa$ (so that any set of cardinality $\kappa$ cannot be cofinal in $\omega_n$). Let $S = \omega_n \times T = [0, \omega_n) \times T$. Then $\beta S$ is a singular compactification.

*Proof*: Let

$$K = [0, \omega_n] \times T = [0, \omega_n) \ \cup \ \{\omega_n\} \ \times \ T = \omega_n + 1 \ \times \ T$$

which densely contains the set, $S = [0, \omega_n) \times T$ and the column $\{\omega_n\} \times T$ (the right edge of $K$). The space $K$ has been shown to be equivalent to $\beta S$ in Theorem 21.27 where $S$ is also seen to be pseudocompact.

Consider the function $f : K \to \{\omega_n\} \times T$, defined as

$$f[[0, \omega_n] \times \{p\}] = (\omega_n, p)$$

[7] Note that the converse fails!

for $p \in T$. The function $f$ can be viewed as the projection map on the compact product space $[0, \omega_n] \times T$ which collapses the $p^{th}$ row onto the point $(\omega_n, p)$. This function is continuous and can be seen as retract of $K$ onto $K \setminus S$. So the Stone Čech compactification of the pseudo-compact space, $S$, is a singular compactification.

---

*Example* 15. Suppose $\alpha\mathbb{R} = \mathbb{R} \cup \{-\infty, \infty\}$ denotes the two-point compactification of $\mathbb{R}$. Show that the function $f : \mathbb{R} \to \alpha\mathbb{R}$ defined as

$$f(x) = x^2 \sin(x)$$

is a singular function on $\mathbb{R}$. Then describe $\mathbb{R} \cup_f S(f)$

*Solution* : See that the function $f(x) = x^2 \sin(x)$ with domain $\mathbb{R}$ has as range $f[\mathbb{R}] = \mathbb{R}$, a dense subset of $\alpha\mathbb{R}$. So $\text{cl}_{\alpha\mathbb{R}} f[\mathbb{R}] = \alpha\mathbb{R}$.

Also, see that if $U$ is an open neighborhood of a point $x$ in $\alpha\mathbb{R}$, then $U \cap (\mathbb{R} \cup \{-\infty, \infty\})$ contains a point $y$ which belongs to $\mathbb{R}$. If $z \in f^{\leftarrow}(y)$, $z > 0$, $f^{\leftarrow}(y) \cap \mathbb{R} \setminus (-z, z)$ is an unbounded subset of $\mathbb{R}$ and so is not compact. So $\text{cl}_{\mathbb{R}}[f^{\leftarrow}[U]$ is not bounded in $\mathbb{R}$ and hence is not compact. Then, by definition, $f$ a singular function on $\mathbb{R}$ which induces the compactification

$$\gamma\mathbb{R} \ = \ \mathbb{R} \cup_f S(f) \ = \ \mathbb{R} \cup_f (\mathbb{R} \cup \{-\infty, \infty\})$$

with outgrowth $S(f) = \mathbb{R} \cup \{-\infty, \infty\}$.[8]

The following example is inspired from the example above.

*Example* 16. Let

$$S = \{(x, y) \in \mathbb{R}^2 : -x^2 \leq y \leq x^2\}$$

viewed as region in $\mathbb{R}^2$, bounded above by $x^2$ and below by $-x^2$. Let $f : S \to \mathbb{R} \cup \{-\infty, \infty\}$ be defined as

$$f(x, y) = y$$

Then $f[S]$ is a dense subset of the two-point compactification $\mathbb{R} \cup \{-\infty, \infty\}$ of $\mathbb{R}$. Then for each $y \in \mathbb{R}$, $f^{\leftarrow}(y)$ is an unbounded subset

$$[\mathbb{R} \times \{y\}] \ \cap \ S$$

[8]Note that the function, $f(x) = x^2 \sin(x)$, also densely embeds $\mathbb{R}$ into the compact space $\beta\mathbb{R}$. Hence, it also generates a singular compactification $\gamma\mathbb{R} = \mathbb{R} \cup^* S(f)$ where $S(f) = \beta\mathbb{R}$.

of the domain $S$. Then, for any open neighborhood $U$ of a point $u$ in $\mathbb{R} \cup \{-\infty, \infty\}$, the subset $\mathrm{cl}_S f^{\leftarrow}[U]$ fails to be compact in $S$. Then $f$ is a singular function on $S$ which induces a singular compactification $\alpha S = S \cup_f S(f)$ with $\mathbb{R} \cup \{-\infty, \infty\}$ as outgrowth. The set $\alpha S$ can then be viewed as the shell of a cylindrically shaped geometrical figure where the extreme right and extreme left edges are sealed together at $\mathbb{R} \cup \{-\infty, \infty\}$.

*Example* 17. For any $a, b \in \mathbb{R}$, the set, $\mathbb{N}$, of natural numbers has a singular compactification, $\alpha\mathbb{N}$, whose outgrowth, $\alpha\mathbb{N}\backslash\mathbb{N}$, is the set $[a, b]$. Hence every bounded closed interval is the continuous image of $\beta\mathbb{N}\backslash\mathbb{N}$.

To see this, simply note that, given that $\mathbb{N}$ is discrete and that the set $\mathbb{Q} \cap [a, b]$ is countable, there is a continuous function, $f : \mathbb{N} \to [a, b]$, which maps $\mathbb{N}$ one-to-one and onto the set $\mathbb{Q} \cap [a, b]$ of all rational number in $[a, b]$. Then $f[\mathbb{N}] = \mathbb{Q} \cap [a, b]$, a dense subset of $[a, b]$; hence $\mathrm{cl}_{[a,b]} f[\mathbb{N}] = [a, b]$.

If $x \in [a, b]$ and $U$ is an open open interval containing $x$ in $[a, b]$, then $x$ is the limit of an infinite sequence of distinct rationals. Then $U$ contains infinitely many distinct elements of $f[\mathbb{N}] = [a, b] \cap \mathbb{Q}$. Then $\mathrm{cl}_{\mathbb{N}} f^{\leftarrow}[U]$ is an infinite subset of $\mathbb{N}$. The set $\mathrm{cl}_{\mathbb{N}} f^{\leftarrow}[U]$ must then be non-compact in $\mathbb{N}$. Then, by definition, $[a, b] = S(f)$.

Since $\mathrm{cl}_{\mathbb{N}} f[U] = S(f)$,

$$\alpha\mathbb{N} = \mathbb{N} \cup_f S(f)$$

is a singular compactification of $\mathbb{N}$ with outgrowth $S(f) = [a, b]$, as claimed.

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**Theorem 22.16** The set, $\mathbb{N}$, of natural numbers has a singular compactification, $\alpha\mathbb{N}$, whose outgrowth, $\alpha\mathbb{N}\backslash\mathbb{N}$, is the set $\beta\mathbb{R}$. Hence $\beta\mathbb{R}$ is a continuous image of $\beta\mathbb{N}\backslash\mathbb{N}$.

*Proof* : Note that, given that $\mathbb{N}$ is discrete and that the set $\mathbb{Q}$ is countable, there is a continuous function, $f : \mathbb{N} \to \beta\mathbb{R}$, which maps $\mathbb{N}$ one-to-one and onto the set, $\mathbb{Q}$, of all rational numbers, a dense subset of $\mathbb{R}$, hence a dense subset of $\beta\mathbb{R}$. Then $\mathrm{cl}_{\beta\mathbb{R}} f[\mathbb{N}] = \mathrm{cl}_{\beta\mathbb{R}}\mathbb{Q} = \beta\mathbb{R}$.

Let $y \in \beta\mathbb{R}$ and $U$ be any $\beta\mathbb{R}$-open neighborhood of $y$. Since $\mathbb{Q}$ is dense in $\beta\mathbb{R}$, then $U \cap \mathbb{Q}$ is an open non-empty subset of $\mathbb{Q}$. Then $U \cap \mathbb{Q} = U \cap f[\mathbb{N}]$ contains an infinite sequence of distinct elements. Given that $f$ is onto $\mathbb{Q}$, $\mathrm{cl}_{\mathbb{N}} f^{\leftarrow}[U]$ is an infinite subset of $\mathbb{N}$ and so must be unbounded in $\mathbb{N}$, hence cannot be compact. By definition, $y \in S(f)$. This holds true independent of the choice of $y$ in $\beta\mathbb{R}$. Then $\beta\mathbb{R} = S(f)$ a singular set of the function $f : \mathbb{N} \to \beta\mathbb{R}$ which densely contains $f[\mathbb{N}] = \mathbb{Q}$. So

$$\alpha\mathbb{N} = \mathbb{N} \cup_f S(f)$$

is a singular compactification of $\mathbb{N}$ with outgrowth $S(f) = \beta\mathbb{R}$, as required.

The function $f : \mathbb{N} \to \beta\mathbb{R}$ extends continuously to $f^\alpha : \alpha\mathbb{N} \to \beta\mathbb{R}$ where $f^\alpha|_{S(f)}$ fixes the points of $S(f) = \beta\mathbb{R}$.

We know that $\alpha\mathbb{N} \preceq \beta\mathbb{N}$. Then the function $\pi_{\beta\to\alpha} : \beta\mathbb{N} \to \alpha\mathbb{N}$ maps the outgrowth $\beta\mathbb{N}\backslash\mathbb{N}$ of $\beta\mathbb{N}$ onto the outgrowth $S(f) = \beta\mathbb{R}$ of $\alpha\mathbb{N}$.

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As a result of the previous theorem, we can also say that $\beta\mathbb{R}$ is a quotient space of $\beta\mathbb{N}\backslash\mathbb{N}$ induced by the continuous map $\pi_{\beta\to\alpha}\beta\mathbb{N} \to \alpha\mathbb{N}$.

*Remark*: Since the cardinality of $\beta\mathbb{N}\backslash\mathbb{N}$ is $2^c$ and $\pi_{\beta\to\alpha}$ maps $\beta\mathbb{N}\backslash\mathbb{N}$ onto $\beta\mathbb{R}$ then this agrees with the arguments presented in the proof of Theorem 21.25 where it is shown that $|\beta\mathbb{R}| = |\beta\mathbb{N}| = 2^c$.

*Example* 18. Consider the continuous function $\sin : \mathbb{N} \to [-1, 1]$. We claim that

$$\mathbb{N} \cup_{\sin} S(\sin) = \mathbb{N} \cup_{\sin} [-1, 1]$$

is a singular compactification of $\mathbb{N}$.

To prove this it suffices to show two things:

($a$) That $\sin[\mathbb{N}]$ is a dense subset of $[-1, 1]$.
($b$) Given that $\sin[\mathbb{N}]$ is a dense subset of $[-1, 1]$ then $S(\sin) = [-1, 1]$, the singular set of the function $\sin : \mathbb{N} \to [-1, 1]$.

The proof of part ($a$) showing that $\sin[\mathbb{N}]$ is a dense subset of $[-1, 1]$ is a bit technical. Certainly, it is no trivial matter, not a fact that should be taken for granted. The proof is deferred to the Appendix.

Part ($b$): To show that $S(\sin) = [-1, 1]$, suppose $x \in [-1, 1]$ and $U$ is an open neighborhood (in $[-1, 1]$) of $x$. Since $\sin[\mathbb{N}]$ is dense in $[-1, 1]$, $U \cap \sin[\mathbb{N}] \neq \varnothing$. Certainly, if $F = U \cap \sin[\mathbb{N}]$, $F$ cannot be finite, since, if it was, $U\backslash F$ would be an open neighborhood of $x$ which misses $\sin[\mathbb{N}]$. So $F$ is infinite. Then $\sin^{\leftarrow}[U] = \cup\{\sin^{\leftarrow}(y) : y \in F\}$ is an infinite pairwise disjoint family of subsets of $\mathbb{N}$. This means that $\sin^{\leftarrow}[U]$ is unbounded in $\mathbb{N}$ and so $\mathrm{cl}_{\mathbb{N}}[\sin^{\leftarrow}[U]]$ cannot be compact.

Then $x \in S(\sin)$. We conclude that $S(\sin) = [-1.1]$.

Then $\sin[\mathbb{N}]$ is a dense subset of $S(\sin) = [-1, 1]$. We conclude that $\mathbb{N} \cup_{\sin} S(\sin) = \mathbb{N} \cup_{\sin} [-1, 1]$ is a singular compactification of $\mathbb{N}$.

*Example* 19. Let $S = (-\pi/2, \pi/2)$ (equipped with the usual topology). Construct a compactification $\alpha S$ of $S$ whose outgrowth is a homeomorphic copy of $[-1, 1]$.

*Solution.* Let $f : (-\pi/2, \pi/2) \to \mathbb{R}$ be defined as,

$$f = \sin \circ \tan$$

Then

$$\begin{aligned} f[(-\pi/2, \pi/2)] &= \sin[\tan[(-\pi/2, \pi/2)]] \\ &= \sin[\mathbb{R}] \\ &= [-1, 1] \end{aligned}$$

Let $x \in (-1, 1)$. Then

$$\begin{aligned} f^{\leftarrow}(x) &= (\sin \circ \tan)^{\leftarrow}(x) \\ &= \arctan[\sin^{\leftarrow}(x)] \end{aligned}$$

where $\sin^{\leftarrow}(x)$ is an unbounded infinite closed subset of $\mathbb{R}$. Then $\arctan[\sin^{\leftarrow}(x)]$ is an infinite closed subset of $(-\pi/2, \pi/2)$. If $y \in [-\pi/2, \pi/2]$ and $U$ is an open neighborhood of $y$ then $\text{cl}_{(-\pi/2,\pi/2)} f^{\leftarrow}[U]$ is not compact in $(-\pi/2, \pi/2)$.

We can then conclude that the function, $\sin \circ \tan : (-\pi/2, \pi/2) \to [-1, 1]$, is a singular map with singular set $S(f) = [-1, 1]$. It induces a singular compactification of $(-\pi/2, \pi/2)$,

$$\alpha(-\pi/2, \pi/2) = (-\pi/2, \pi/2) \cup_f S(\sin \circ \tan)$$

So $\alpha S = \alpha(-\pi/2, \pi/2)$ is a compact space such that $\text{cl}_{\alpha S} S \backslash S$ is homeomorphic to $[-1, 1]$.

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## Concepts review.

1. Given a continuous function $f : S \to T$ from the completely regular set $S$ into the compact set $T$, define the singular set $S(f)$.

2. Given a continuous function $f : S \to T$ from the completely regular set $S$ into the compact set $T$, what does it mean to say that $f$ is a singular map?

3. Given a continuous function $f : S \to T$ from the completely regular set $S$ into the compact set $T$, define a compactification induced by $f$.

4. What does it mean to say that a subset $T$ of a set $S$ is a *retract* of $S$?

5. Given a continuous function $f : S \to T$ from the completely regular set $S$ into the compact set $T$, define a singular compactification induced by $f$.

6. Describe the topology of on a non-singular compactification $S \cup^* S(f)$.

7. Show that arctan: $\mathbb{R} \to [-\pi/2, \pi/2]$ is not a singular map. Find a compactification of $\mathbb{R}$ induced by arctan.

8. Given a continuous function $f : S \to K$ mapping a locally compact completely regular set $S$ into a compact set $K$ and its extension $f^{\beta(K)} : \beta S \to K$ express the singular set $S(f)$ in terms of $f^{\beta(K)}$.

9. Given a non-singular compactification $\alpha S = S \cup^* S(g)$ in what way does the function $g : S \to K$ extend to $g^\alpha : \alpha S \to K$.

10. Produce an example of a pair of singular compactifications with the same singular sets but which are not equivalent.

11. Produce an example of a Stone-Čech compactification which is not singular.

12. State one way of recognizing a pair of singular compactifications which are equivalent.

13. Justify the statement "$\mathbb{R} \cup_{\sin} S(\sin)$ is a singular compactification of $\mathbb{R}$. Describe its outgrowth.