21 / Hausdorff compactifications.

Abstract. *In this section, we discuss those spaces, S, which can be densely embedded in a compact Hausdorff space. Only completely regular spaces can possess this property. The process by which we determine such a compact space, αS, for S, is called compactifying S. The space, αS, is called the compactification of S. The family of all compactifications of a completely regular space can be partially ordered. The maximal compactification of S with respect to the chosen partial ordering is called the Stone-Čech compactification. We discuss methods for its construction. We will show that only locally compact spaces have a minimal compactification with respect to the chosen partial ordering. It is called the one-point compactification. We present a few properties of $\beta\mathbb{R}$, $\beta\mathbb{Q}$ and $\beta\mathbb{N}$. In particular we show that $\beta\mathbb{N}$ is not sequentially compact. This chapter is of interest to readers who would like the opportunity to practice applying, in an interesting context, most of the concepts studied in the first part of* Point-set topology with topics.

21.1 Compactifying a space

In this section we will briefly talk about methods for "compactifying a space (S, τ_S)". This essentially means adding a set, F, of points to S, to obtain a larger set, $T = S \cup F$, and topologizing T so that (T, τ_T) is a compact Hausdorff space in which a homeomorphic copy of S appears as a dense subspace of T.

With certain bounded subspaces of $\mathbb{R}^n$, this can, sometimes, be quite easy to do. For example, if $S = [-1, 3) \cup (3, 7)$ is equipped with the subspace topology, then by simply adding the points $\{3, 7\}$ to S we obtain the set $T = S \cup \{3, 7\} = [-1, 7]$ which, when equipped with the subspace topology, is a compact Hausdorff space which densely contains a homeomorphic copy of S. In such a case, we will say that T is a *compactification* of S. A compactification of a bounded subset of $\mathbb{R}^n$ can always be obtained by taking its closure. This does not mean, however, that there are not others. If we are given a space such as $\mathbb{N}$ or $\mathbb{Q}$, it is not at all obvious how one would go about compactifying such spaces. We will show techniques which allow us to achieve this objective.

For what follows, recall that the evaluation map $e : S \to \prod_{i\in J}[a_i, b_i]$ induced by $C^*(S)$ (the set of all continuous bounded real-valued functions on S) is defined as

$$e(x) = <f_i(x)>_{i\in J} \in \prod_{i\in J}[a_i, b_i]$$

where $f_i \in C^*(S)$ and $f_i[S] \subseteq [a_i, b_i]$.

Definition 21.1 Let (S, τ_S) be a topological space and (T, τ_T) be a compact Hausdorff space. We will say that T *is a compactification of* S if S is densely embedded in T.[1]

If S is a compact Hausdorff space, then S can be viewed as being a compactification of itself. Recall that a compact Hausdorff space is normal, and so is completely regular. Since subspaces of completely regular spaces are completely regular, then

... only a completely regular space can have a compactification.

In the *Embedding theorem III*, Theorem 14.7, we showed how any completely regular space can be compactified. The evaluation map, $e : S \to T$, induced by $C^*(S)$ embeds S into a cube,

$$T = \prod_{i\in J}[a_i, b_i]$$

Since each interval $[a_i, b_i]$ is homeomorphic to $[0, 1]$, then there is a homeomorphism, $h : T \to \prod_{i\in J}[0, 1]$, which maps T onto $P = \prod_{i\in J}[0, 1]$. By Tychonoff's theorem, P is guaranteed to be compact. So the function, $q : S \to P$, defined as, $q = h\circ e$, embeds S into $\prod_{i\in J}[0, 1]$. Hence $\text{cl}_P q[S]$ is a compact subspace of the product space, P, which densely contains the homeomorphic image, $q[S]$, of S.[2] So, even common topological spaces such as $\mathbb{R}$, $\mathbb{Q}$, and $\mathbb{N}$ have at least the compactification obtained by the method just described.

[1] When we say "compactification of S" we always mean a "Hausdorff compactification", unless explicitly stated otherwise.

[2] This is just one small example which shows why Tychonoff theorem deserves to be titled and why it is such an important theorem in topology.

21.2 The Stone-Čech compactification.

We have described only one of the various ways to obtain a homeomorphic copy of the compactification, $\mathrm{cl}_T e[S]$, of S. This particular compactification has a special name.

Definition 21.2 Let (S, τ_S) be a completely regular topological space. Let

$$e : S \to \prod_{i \in I} [a_i, b_i]$$

be the evaluation map induced by $C^*(S)$ which embeds S in the *compact* product space, $T = \prod_{i \in I} [a_i, b_i]$. Then the function e densely embeds S into the compact space $\mathrm{cl}_T e[S]$. So $\mathrm{cl}_T e[S]$ satisfies the definition of a compactification of S.

The subspace,

$$\mathrm{cl}_T e[S]$$

is called the *Stone-Čech compactification of S*. The Stone-Čech compactification of S is uniquely (and universally) denoted by, βS. We can then write,

$$\beta S = \mathrm{cl}_T e[S]$$

It is important to remember that, complete regularity of S, is all that is required to construct the Hausdorff compactification βS. The space S need not be locally compact. Hence, if a space S is not locally compact, we still obtain a Hausdorff compactification. Recall that Corollary 18.4 states that, " ... if S is dense in a Hausdorff compact set K, then S is locally compact if and only if S is open in K". Since $\beta\mathbb{Q}$ is a Hausdorff compactification of $\mathbb{Q}$ and $\mathbb{Q}$ is not locally compact then $\mathbb{Q}$ cannot be open in $\beta\mathbb{Q}$.

Equivalent compactifications

Given a completely regular space S, there is nothing stated up to now which would lead us to conclude that S has only one compactification. In fact most spaces we will consider will have many compactifications. Suppose we are given two compactifications for a space, S, say αS and γS. If there is a homeomorphism

$$h : \alpha S \to \gamma S$$

mapping αS onto γS such that $h(x) = x$ for all $x \in S$, then αS and γS will be considered to be *equivalent compactifications* of S. Two compactifications of the same space which have been determined to be "equivalent" in this way will be assumed to be the same compactification, topologically speaking. This equivalence is often expressed by the symbol,

$$\alpha S \equiv \gamma S$$

For convenience, some authors express equivalence of two compactification by "$\alpha S = \gamma S$" even though, strictly speaking, αS and γS may not necessarily be equal sets. But one should be cautious! There are compactifications αS and γS, of a space S such that $\alpha S = \gamma S$, but their respective topologies are such that $\alpha S \not\equiv \gamma S$. Such examples are presented on pages 532 and 544.

So if αS is any compactification of S for which there is a homeomorphism $h : \alpha S \to \beta S$ which fixes the points of S αS is said to be equivalent to the βS.

The outgrowth of a topological space.

Given a topological space, S, and a compactification, αS, the set

$$\alpha S \backslash S$$

is referred to as being the *outgrowth of* S with respect to this particular compactification or as the *remainder* of αS. Equivalent compactifications will be considered to have the same outgrowth. We will sometimes be interested in determining whether an outgrowth satisfies certain topological properties.

Example 1 : *Compactifications of* $\mathbb{R}$. In this example we construct two compactifications of $\mathbb{R}$.

A compactification of $\mathbb{R}$ *with two points in its outgrowth.* We know that $\arctan : \mathbb{R} \to (-\pi/2, \pi/2)$ maps $\mathbb{R}$ homeomorphically onto $(-\pi/2, \pi/2)$. Hence $\mathbb{R}$ is embedded in the compact space

$$\gamma\mathbb{R} = \arctan[\mathbb{R}] \cup \{-\pi/2, \pi/2\} = [\pi/2, \pi/2]$$

with outgrowth $\gamma\mathbb{R}\backslash\mathbb{R} = \{-\pi/2, \pi/2\}$. We showed that arctan densely embeds a copy of $\mathbb{R}$ in the compact set $[-\pi/2, \pi/2]$. This compactification is often simply expressed as

$$\gamma\mathbb{R} = \mathbb{R} \cup \{-\pi/2, \pi/2\}$$

provided it is understood that arctan is the embedding function. (See the example on page 530 where we discuss this compactification from a different perspective.)

We provide another way to visualize a two-point compactification of $\mathbb{R}$. Suppose $K = \mathbb{R} \cup \{-\infty, \infty\}$ is a set where $-\infty$ and ∞ represents points which don't belong to $\mathbb{R}$. Let $\arctan^K : K \to [-\pi/2, \pi/2]$ be a function mapping K onto $[-\pi/2, \pi/2]$ (equipped with the usual topology) be defined as

$$\begin{aligned} \arctan^K|_{\mathbb{R}} &= \arctan \\ \arctan^K(-\infty) &= -\pi/2 \\ \arctan^K(\infty) &= \pi/2 \end{aligned}$$

We can topologize K by declaring that U is open in K if and only if $U = \arctan^{K\leftarrow}[V]$ for some open V in $[-\pi/2, \pi/2]$. Then τ_K is the weak topology on K induced by the function $\arctan^K$. The subspace topology on $\mathbb{R}$ is the usual topology. Then B_1 is an open neighborhood of $-\infty$ if and only if $U = \arctan^{K\leftarrow}[V]$ for some open neighborhood of $-\pi/2$ and B_2 is an open neighborhood of ∞ if and only if $U = \arctan^{K\leftarrow}[V]$ for some open neighborhood of $\pi/2$. Then, with this topology, K can be viewed as a two-point compactification

$$\delta\mathbb{R} = \mathbb{R} \cup \{-\infty, \infty\}$$

This is often a preferred way to visualize a two-point compactification of $\mathbb{R}$.

A compactification of $\mathbb{R}$ *with a single point in its outgrowth.* We know that there is a homeomorphism $h : \mathbb{R} \to (0, 2\pi)$ which maps $\mathbb{R}$ onto $(0, 2\pi)$. The function,

$$h(x) = 2[\arctan(x) + \pi/2]$$

is an example. Define the homeomorphic function $f : (0, 2\pi) \to \mathbb{R}^2$ as

$$f(x) = (\sin(x), \cos(x))$$

Then $f \circ h : \mathbb{R} \to \mathbb{R}^2$ densely embeds $\mathbb{R}$ into the (closed and bounded) compact set

$$\alpha\mathbb{R} = K = \{(\sin(x), \cos(x)) : x \in (0, 2\pi)\} \ \cup \ \{(1, 0)\}$$

with the single point $(1, 0)$ in its outgrowth, $\alpha\mathbb{R}\backslash\mathbb{R} = \{(1, 0)\}$.

We will say more about a compactification which has a single point in its outgrowth in Theorem 21.13.

Example 2. Let $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$. Is there a compactification of $\mathbb{R}^+$ which contains only two points?

Solution: No. Suppose $\alpha\mathbb{R}^+ = \mathbb{R}^+ \cup \{a, b\}$. Since $\{a, b\}$ is a compact subset of $\alpha\mathbb{R}^+$ and $\mathbb{R}^+$ is dense in $\alpha\mathbb{R}^+$ any connected open neighborhood, say B, of $\{a, b\}$ intersects $\mathbb{R}^+$. Then $\alpha\mathbb{R}\backslash B$ is a compact set, say $[0, k]$. Then, since $\alpha\mathbb{R}^+$ is Hausdorff, there must be disjoint open neighborhoods B_1 and B_2 in $\alpha\mathbb{R}^+$ of a and b, respectively, such that $B = (B_1 \cup B_2) \cap \mathbb{R}^+$. Then $(B_1 \cup B_2) \cap \mathbb{R}^+ = (k, \infty)$. Since B_1 and B_2 do not intersect in $\mathbb{R}^+$ and (k, ∞) is connected we have a contradiction. So $\mathbb{R}^+$ cannot have an outgrowth containing only two points.

21.3 A partial ordering of Hausdorff compactifications of a space.

Let's gather together all Hausdorff compactifications of a completely regular space, S, as follows,

$$\mathscr{C} = \{\alpha_i S : i \in I\}$$

We will partially order the family, $\mathscr{C}$, by defining "$\prec$" in the following way:

> *If $\alpha_i S$ and $\alpha_j S$ belong to $\mathscr{C}$, we will write*
>
> $$\alpha_i S \prec \alpha_j S$$
>
> *if and only if there is a continuous function $f : \alpha_j S \to \alpha_i S$, not one-to-one, mapping $\alpha_j S\backslash S$ onto $\alpha_i S\backslash S$ which fixes the points of S. If $f : \alpha_j S \to \alpha_i S$ is a homeomorphism we communicate this by writing $\alpha_i S \equiv \alpha_j S$ or verbally stating that are $\alpha_i S$ and $\alpha_j S$ are equivalent compactifications.*

If $\alpha S \prec \gamma S$ we will express the continuous function, $f : \gamma S \to \alpha S$, as

$$\pi_{\gamma\to\alpha} : \gamma S \to \alpha S$$

The function $\pi_{\gamma\to\alpha}$ explicitly expresses which compactification is strictly larger than the other. If αS and γS are such that neither is strictly less than the other and are not equivalent compactifications, then we say that αS and γS are not comparable in $\mathscr{C}$. We will see pairs of compactifications of a space which are not comparable later in the text.

The expression

$$\alpha S \preceq \gamma S$$

is to be interpreted as "$\alpha S \prec \gamma S$ or $\alpha S \equiv \gamma S$".

Remark. Observe that given compactifications γS and αS such that $\alpha S \preceq \gamma S$, the function $\pi_{\gamma\to\alpha} : \gamma S \to \alpha S$ maps a compact set onto a compact set and so is a closed function. Then the topology on αS can be seen as a quotient topology induced by the quotient map $\pi_{\gamma\to\alpha}$. (See Theorem 8.2)

21.4 On C^*-embedded subsets of a topological space.

Given a subset T of a topological space, S, and a continuous bounded real-valued function $f : T \to \mathbb{R}$, it is not guaranteed that there is a bounded continuous function $g : S \to \mathbb{R}$ such that $g|_T = f$ on T. If there is, then we will say that

"*g is a continuous extension of f from T to S*"

This motivates the following definition.

Definition 21.3 Let (S, τ) be a topological space and U be a proper non-empty subset of S. We say that U is *C^*-embedded* in S if every real-valued bounded function, $f \in C^*(U)$, extends to a function, $g \in C^*(S)$, in the sense that $g|_U = f$.[3] In this case we will say that...

$g : S \to \mathbb{R}$ is a *continuous extension* of $f : U \to \mathbb{R}$ from U to S

The notion of "C^*-embedding" is closely related to completely regular spaces and their Stone-Čech compactification. For this reason, we will discuss this property in depth now (even though C^*-embeddings may be discussed in other contexts). The following theorem shows that, for any completely regular space S, S is C^*-embedded in its Stone-Čech compactification, βS. That is, *every function, $f \in C^*(S)$, extends continuously to a function, $f^\beta \in C(\beta S)$*

Theorem 21.4 Let S be a completely regular space. Then S is C^*-embedded in βS.

[3] There is an analogous definition for "C-embedded" studied later: We say that U is *C-embedded* in S if every real-valued function, $f \in C(U)$, continuously extends to a function, $f^* \in C(S)$

Proof: If $f \in C^*(S)$, let I_f be the range of f. Let

$$T = \prod_{f \in C^*(S)} \text{cl}_{\mathbb{R}} I_f$$

Then the evaluation map, $e : S \to T$ embeds S in T. Recall that, by definition,

$$\beta S = \text{cl}_T e[S] \subseteq \prod_{f \in C^*(S)} \text{cl}_{\mathbb{R}} I_f$$

Suppose $g \in C^*(S)$.

We are required to show that $g : S \to \mathbb{R}$ extends continuously to some function, $g^\beta : \beta S \to \mathbb{R}$.

If π_g is the g^{th}-projection map, then

$$\pi_g : \prod_{f \in C^*(S)} \text{cl}_{\mathbb{R}} I_f \to \text{cl}_{\mathbb{R}} I_g$$

where $\beta S = \text{cl}_T e[S]$ and so,

$$\pi_g|_{\beta S} : \beta S \to \text{cl}_{\mathbb{R}} I_g$$

maps βS into $\text{cl}_{\mathbb{R}} I_g$. Let $g^\beta = \pi_g|_{\beta S}$. Then $g^\beta[\beta S] = g^\beta[\text{cl}_T e[S]] = \text{cl}_{\mathbb{R}} g[S] \subseteq \text{cl}_{\mathbb{R}} I_g$.

It follows that $g^\beta : \beta S \to \text{cl}_{\mathbb{R}} I_g$ and, since $g[S] \subseteq I_g$, for $x \in S$, $g^\beta|_S(x) = g(x)$.

So g^β is a continuous extension of g from S to βS.

Note that, in the case where S is a compact space, $e[S]$ is a compact space densely embedded in βS and so $\beta S \backslash S = \varnothing$. Then S and βS are homeomorphic.

The theorem guarantees that every real-valued continuous bounded function, f, on a completely regular space, S, extends to a continuous function $f^\beta : \beta S \to \mathbb{R}$. The function f^β is the *extension of f* from S to βS. Recall (from Theorem 9.8) that continuity guarantees that two continuous functions which agree on a dense subset D of a Hausdorff space, S, must agree on all of S. So there can only be one extension, f^β, of f from S to βS.

We will soon show (in Theorem 21.9) that, if αS is a compactification of S and S is C^*-embedded in αS, then αS must be the compactification, βS. That is,

...βS is the only compactification in which S is C^*-embedded

A generalization of the extension, $f \to f^\beta$.

The above theorem can be generalized a step further. Suppose

$$C(S, K)$$

denotes all continuous functions mapping S into a compact set K. We show that every function f in $C(S, K)$ extends to a function $f^{\beta(K)} \in C(\beta S, K)$. Note that neither f nor $f^{\beta(K)}$ need be real-valued. The space, K, represents any compact set which contains the image of S under f. We present now the following very important theorem.

Theorem 21.5 Let S be a completely regular (non-compact) space and $g : S \to K$ be a continuous function mapping S into a compact Hausdorff space, K. Then g extends uniquely to a continuous function, $g^{\beta(K)} : \beta S \to K$.

Proof: We are given that $g : S \to K$ continuously maps the completely regular space, S, into the compact Hausdorff space K. Since S is completely regular it has a compactification, βS.

We are required to show that g extends to $g^{\beta(K)} : \beta S \to K$.

Since K is compact Hausdorff it is completely regular; hence, if $\mathscr{K} = C(K)$, the evaluation map

$$e_{\mathscr{K}} : K \to \prod_{k \in \mathscr{K}} [0, 1]_k$$

embeds K in the product space $\prod_{k \in \mathscr{K}} [0, 1]_k$ where $e_{\mathscr{K}}[g[S]] \subseteq e_{\mathscr{K}}[K]$ and

$$e_{\mathscr{K}}[g[S]] = < k\,[g[S]] >_{k \in \mathscr{K}} \subseteq \prod_{k \in \mathscr{K}} [0, 1]_k$$

We also have,

$$g(x) = e_{\mathscr{K}}^{\leftarrow}(e_{\mathscr{K}}(g(x))) = e_{\mathscr{K}}^{\leftarrow}(< (k \circ g)(x)) >_{k \in \mathscr{K}})$$

Since $(k \circ g) : S \to [0, 1]_k$ extends to $(k \circ g)^\beta : \beta S \to [0, 1]_k$, for each $k \in \mathscr{K}$, we define a continuous function $h : \beta S \to K$ as,

$$h(x) = e_{\mathscr{K}}^{\leftarrow}(< (k \circ g)^\beta(x) >_{k \in \mathscr{K}})$$

By defining, $g^{\beta(K)} : \beta S \to K$ as,

$$g^{\beta(K)}(x) = h(x)$$

we obtain $h(x)$ as an extension of $g : S \to K$ to the continuous function,

$$g^{\beta(K)} : \beta S \to K$$

Given a completely regular topological space S, and $T = \prod_{i\in I}[a_i, b_i]$ we now see that the evaluation function $e_{C^*(S)} : S \to \prod_{i\in I}[a_i, b_i]$ (which homeomorphically embeds a copy of S into T) then continuously extends to βS as follows:

$$\begin{aligned} e^{\beta}_{C^*(S)}[\beta S] &= e^{\beta}_{C^*(S)}[\mathrm{cl}_{\beta S} S] \\ &= \mathrm{cl}_T e_{C^*(S)}[S] \end{aligned}$$

where

$$e^{\beta}_{C^*(S)}(x) = < f^{\beta}(x) >_{f\in C^*(S)}$$

The Stone-Čech compactification, βS, is then equivalent to $e^{\beta}_{C^*(S)}[\beta S]$.

The maximal compactification, βS.

Suppose αS is any compactification of S possibly distinct from βS. We will now show that, in the partially ordered family, $\mathscr{C}$, of all *Hausdorff* compactifications of S, $\alpha S \preceq \beta S$. Showing this requires that we produce a continuous function $\pi_{\beta\to\alpha} : \beta S \to \alpha S$ such that $\pi_{\beta\to\alpha}(x) = x$, for all $x \in S$. If we can prove this, then we will have shown that βS is the unique maximal compactification of a completely regular space, S.

Theorem 21.6 If αS is a compactification of S, then $\alpha S \preceq \beta S$.

Proof: By Theorem 21.5, the identity map $i : S \to \alpha S$, extends to a continuous function,

$$i^* : \beta S \to \alpha S$$

Then $S \subseteq i^*[\beta S] \subseteq \alpha S$, where $i^*[\beta S]$ is compact, hence closed in αS. Since S is dense in αS, then the open set $\alpha S \backslash i^*[\beta S]$ must be empty. So the continuous function,

$$i^*[\beta S] = i^*[\mathrm{cl}_{\beta S} S] = \mathrm{cl}_{\alpha S} i^*[S] = \mathrm{cl}_{\alpha S} S = \alpha S$$

maps βS onto αS. So $\alpha S \preceq \beta S$, as required.

Then, for any compactification αS, there is the continuous function

$$\pi_{\beta\to\alpha} : \beta S \to \alpha S$$

which maps βS onto αS where $\pi_{\beta\to\alpha}$ fixes the points of S.

More on $\beta\mathbb{R}$.

Since $\mathbb{R}$ is connected then, since $\mathbb{R}$ is dense in $\beta\mathbb{R}$, $\beta\mathbb{R}$ is connected. Suppose $\alpha\mathbb{R} = \mathbb{R} \cup \{p_1, p_2\}$ denotes the two-point compactification of $\mathbb{R}$. Then, by the above theorem, $\alpha\mathbb{R} \preceq \beta\mathbb{R}$. The function, $\pi_{\beta\to\alpha} : \beta\mathbb{R} \to \alpha\mathbb{R}$, then continuously maps $\beta\mathbb{R}\backslash\mathbb{R}$ onto $\{p_1, p_2\}$. So $\beta\mathbb{R}\backslash\mathbb{R}$ is the disjoint union of the two compact subsets, ${\pi_{\beta\to\alpha}}^{\leftarrow}(p_1)$ and ${\pi_{\beta\to\alpha}}^{\leftarrow}(p_2)$. We conclude that $\beta\mathbb{R}\backslash\mathbb{R}$ is *not* a connected subspace of $\beta\mathbb{R}$.

21.5 More on C^*-embedded subsets of a space S.

We present a miscellany of results which will help us more easily recognize a C^*-embedded subset, T of a space S. We will return to our discussion of compactification immediately following this.

The simplest example of a C^*-embedded subset of $\mathbb{R}$ is a compact subset of $\mathbb{R}$.

Theorem 21.7 If K is a compact subset of $\mathbb{R}$, then K is C^*-embedded in $\mathbb{R}$.

Proof: Let K be a compact subset of $\mathbb{R}$ and $f : K \to \mathbb{R}$ be a continuous function on K. Since every continuous real-valued function is bounded on a compact subset, then $f \in C^*(K) = C(K)$. Suppose

$$u = \sup\{K\} \text{ and } v = \inf\{K\}$$

Since K is closed and bounded in $\mathbb{R}$, then u and v belong to K. Suppose $g : \mathbb{R} \to \mathbb{R}$ is a function such that

$$\begin{aligned} g &= f \text{ on } K \\ g(x) &= f(u), \text{ if } x \geq u \\ g(x) &= f(v), \text{ if } x \leq v \end{aligned}$$

It is easily verified that g is a continuous extension of f from K to $\mathbb{R}$. Then K is C^*-embedded in $\mathbb{R}$.

Example 3. The set $\mathbb{N}$ is C^*-embedded in $\mathbb{R}$. One way of visualizing this is to plot the points of $\{(n, f(n)) : n \in \mathbb{N}\}$ of a function $f \in C^*(\mathbb{N})$ in the Cartesian plane $\mathbb{R}^2$ and join every pair of successive points $(n, f(n))$ and $(n+1, f(n+1))$ by a straight line. This results in a continuous curve representing a continuous function g on $\mathbb{R}$ which extends f.

Urysohn's extension theorem.

It is usually not easy to determine whether a subset T in a space S is C^*-embedded in S. The following theorem often referred to as *Urysohn's extension theorem* provides an important and useful tool for recognizing C^*-embedded sets.[4] Many view this theorem as one of the most important theorems in general topology.

Theorem 21.8 *Urysohn's extension theorem:* Let T be a subset of the completely regular space S. Then T is C^*-embedded in S if and only if pairs of sets which can be completely separated by some function in $C^*(T)$ can also be separated by some function in $C^*(S)$.

Proof: ($\Rightarrow$) Suppose T is C^*-embedded in S and U and V are completely separated subsets of T. Then there exists $f \in C^*(T)$ such that $U \subseteq f^{\leftarrow}(0)$ and $V \subseteq f^{\leftarrow}(1)$. Then, by hypothesis, f extends to $f^* \in C^*(S)$. Then $U \subseteq f^{\leftarrow *}(0)$ and $V \subseteq f^{\leftarrow *}(1)$. So U and V are completely separated in S.

($\Leftarrow$) Suppose that pairs of sets which can be completely separated by some function in $C^*(T)$ can also be separated by some function in $C^*(S)$.

Let f_1 be a function in $C^*(T)$. We are required to show that there exists a function $g \in C^*(S)$ such that $g|_T = f_1$.

Since f_1 is bounded on T, then there exists, $k \in \mathbb{R}$, such that $|f_1(x)| \leq k$ for all $x \in T$. Then $f_1 \leq k = 3r_1 = 3 \cdot \left[\frac{k}{2} \cdot \left(\frac{2}{3}\right)^1\right] = k$ (Where $r_1 = \frac{k}{2} \cdot \left(\frac{2}{3}\right)^1$)

[4]The Urysohn's extension theorem should not be confused with the *Urysohn's lemma.* Urysohn's lemma states that "The topological space (S, τ_S) is *normal* if and only if given a pair of disjoint non-empty closed sets, F and W, in S there exists a continuous function $f : S \to [0,1]$ such that, $F \subseteq f^{\leftarrow}[\{0\}]$ and $W \subseteq f^{\leftarrow}[\{1\}]$"

We now inductively define a sequence of functions $\{f_n\} \subseteq C^*(T)$. For $n = 1, 2, 3, \ldots$, there exists $f_n \in C^*(T)$ such that $-3r_n \leq f_n(x) \leq 3r_n$ where,

$$3r_n = 3 \cdot \left[\frac{k}{2} \cdot \left(\frac{2}{3}\right)^n\right] = k \cdot \left(\frac{2}{3}\right)^{n-1} \quad \text{(Where } r_n = \tfrac{k}{2} \cdot \left(\tfrac{2}{3}\right)^n\text{)}$$

For this n, let $U_n = f^{\leftarrow}{}_n[\,[-3r_n, -r_n]\,]$ and $V_n = f^{\leftarrow}{}_n[\,[r_n, 3r_n]\,]$.

We see that U_n and V_n are completely separated in T. [5]

By hypothesis, U_n and V_n are completely separated in S. This means there exists $g_n \in C^*(S)$ such that, $g_n[S] \subseteq [-r_n, r_n]$, $g_n[U_n] = \{-r_n\}$ and $g_n[V_n] = \{r_n\}$
(where $r_n = \left[\frac{k}{2} \cdot \left(\frac{2}{3}\right)^n\right]$). So the sequence, $\{g_n : n = 1, 2, 3, \ldots\}$, thus constructed is well-defined in $C^*(S)$.

We now inductively define the sequence $\{h_n\} \subseteq C^*(T)$ initiating the process with $h_1 = f_1$ and continuing with

$$h_{n+1} = h_n - g_n|_T$$

Then for each n,

$$|h_{n+1}| \leq 2r_1 = 2 \cdot \frac{k}{2} \cdot \left(\frac{2}{3}\right)^n = 3 \cdot \frac{k}{2} \cdot \left(\frac{2}{3}\right)^{n+1} = 3r_{n+1}$$

So $g_n|_T = h_n - h_{n+1}$. Define $g : S \to \mathbb{R}$ as the series

$$g(x) = \sum_{n \in \mathbb{N} \setminus \{0\}} g_n(x)$$

See that $g(x)$ is continuous on S: Since $|g_n(x)| \leq \frac{k}{2}\left(\frac{2}{3}\right)^n$, and $\sum_{n \in \mathbb{N} \setminus \{0\}} \frac{k}{2}\left(\frac{2}{3}\right)^n$ is a converging geometric series, then $\sum_{n \in \mathbb{N} \setminus \{0\}} g_n(x)$ converges uniformly to $g(x)$. Since each $g_n(x)$ is continuous on S, then $g \in C^*(S)$. So g is a continuous on S.

Also, see that, $g|_T = f_1$:

$$\begin{aligned}
g|_T(x) &= \lim_{m \to \infty} \sum_{n=1}^{m} g_n|_T(x) \\
&= \lim_{m \to \infty} (h_1(x) - h_2(x)) + (h_2(x) - h_3(x)) + \cdots + (h_m(x) - h_{m+1}(x)) \\
&= \lim_{m \to \infty} h_1(x) - h_{m+1}(x) \\
&= h_1(x) = f_1(x) \quad \text{(Since } \lim_{m \to \infty} 3r_{m+1} = 0\text{)}
\end{aligned}$$

[5] To see this: the function $h_n = (-r_n \vee f_n) \wedge r_n)$ has $U_n \subseteq Z(h_n - (-r_n))$ and $V_n \subseteq Z(h_n - r_n)$.

Then, $g|_T = f_1$ so f_1 extends continuously from T to S. We are done.

Suppose F and K are disjoint compact subsets in a completely regular space, S. Then F and K are disjoint closed subsets of βS. Since βS is compact Hausdorff then it is normal. Then there is a function f in $C(\beta S)$ which separates F and K. So $f|_S$ separates F and K in S. We have just proven the following useful statement:

> "*Disjoint compact subsets of a completely regular space are completely separated.*"

Example 4. Use Urysohn's extension lemma to show that any compact subset, K, of a completely regular space, S, is C^*-embedded in S.

Solution: Let U and V be disjoint subsets of the compact set, K, which are completely separated in K. Then there is a function $f \in C(K)$ such that $U \subseteq A = f^{\leftarrow}[\{0\}]$ and $V \subseteq B = f^{\leftarrow}[\{1\}]$. Both A and B are disjoint closed subsets of compact K and so are compact sets. Then A and B are compact in the completely regular space, S. Then A and B are completely separated in S. So U and V are completely separated in S. By Urysohn's extension lemma, K is C^*-embedded in S. We conclude that ...

> ... *any compact subset, K, of a completely regular space, S, is C^*-embedded in S.*

Uniqueness of βS.

We are now able to prove that, up to equivalence, the Stone-Čech compactification of S is the only compactification in which S is C^*-embedded. By this we mean that, if S is C^*-embedded in the compactification, γS, of S, then γS is equivalent to βS. So the symbol, βS, is strictly reserved for the Stone-Čech compactification of S.

Theorem 21.9 The completely regular space S is C^*-embedded in the compactification, γS, if and only if $\gamma S \equiv \beta S$.

Proof: We are given that S is completely regular.

($\Leftarrow$) Suppose $\gamma S \equiv \beta S$. Then there is a homeomorphism, $h : \gamma S \rightarrow \beta S$, mapping γS onto βS such that $h(x) = x$, for all $x \in S$. We are required to show that S is C^*-embedded in γS.

If $f \in C^*(S)$, then f extends to $f^\beta : \beta S \to \mathbb{R}$. Then $f^\beta \circ h : \gamma S \to \mathbb{R}$. Define $f^\gamma = f^\beta \circ h$. We see that $f^\gamma : \gamma S \to \mathbb{R}$ is the continuous extension of f from S to γS. Then S is C^*-embedded in γS.

($\Rightarrow$) Suppose S is C^*-embedded in γS. We are required to show that γS and βS are equivalent compactifications.

Let $i : S \to S$ be the identity map. Then, by Theorem 21.5, i extends to $i^* : \beta S \to \gamma S$. Also, just as shown in the proof of theorem 21.5, $i^\leftarrow$ extends to $i^{\leftarrow\wedge} : \gamma S \to \beta S$. Then $i^\leftarrow \circ i$ and $i \circ i^\leftarrow$ are both identity maps on S and, since S is dense in both βS and γS, respectively, then $i^* \circ i^{\leftarrow\wedge}$ and $i^{\leftarrow\wedge} \circ i^*$ are identity maps on γS and on βS, respectively. Then both i^* and $i^{\leftarrow\wedge}$ are homeomorphisms. Hence γS and βS are equivalent compactifications.

Example 5. Show that, if F is a closed subset of a metric space S, then F is C^*-embedded in S.

Solution: Let F be a closed subset of the metric space S. We will set up the solution so that we can invoke the Urysohn extension theorem.

Let A and B be completely separated in F. Then, by definition, there is a function f in $C^*(F)$ such that $A \subseteq Z(f)$ and $B \subseteq Z(f-1)$. Then $\text{cl}_F A \subseteq Z(f)$ and $\text{cl}_F B \subseteq Z(f-1)$. Since F is closed in S, then so are the disjoint sets $\text{cl}_F A$ and $\text{cl}_F B$. It is shown on page 243 that in metric spaces closed subsets are zero-sets. So $\text{cl}_F A$ and $\text{cl}_F B$ are disjoint zero-sets in S, say $\text{cl}_F A = Z(k)$ and $\text{cl}_F B = Z(g)$ in S. If

$$h = \frac{|k|}{|k| + |g|}$$

on S, $\text{cl}_F A = Z(h)$ and $\text{cl}_F B = Z(h-1)$ in S. So A and B are completely separated in S. By Urysohn's extension lemma every closed subset of a metric space is C^*-embedded.

Because of this, it is useful to remember that,

. . . any closed subset of $\mathbb{R}$ is C^*-embedded in $\mathbb{R}$

For example, $\mathbb{N}$ is C^*-embedded in $\mathbb{R}$, since it is closed in $\mathbb{R}$ (its complement being open in $\mathbb{R}$). However, $\mathbb{Q}$ is not C^*-embedded in $\mathbb{R}$. To see this, note that, if $f(x) = \sin \frac{1}{x-\pi}$, the function $f|_{\mathbb{Q}}(x)$ is a bounded continuous function on $\mathbb{Q}$ which does not extend to $\mathbb{R}$.

21.6 Topic: Associating compactifications to subalgebras of $C^*(S)$.

Recall the theorem statement titled The embedding theorem I (Theorem 7.16). It guarantees that:

> "If $\mathscr{F}$ is a subset of $C^*(S)$ which separates points and closed sets of S, the evaluation map, $e_{\mathscr{F}} : S \to \prod_{f\in\mathscr{F}}[a_f, b_f]$, maps S homeomorphically into the compact set $T = \prod_{f\in\mathscr{F}}[a_f, b_f]$, Then $\text{cl}_T e_{\mathscr{F}}[S]$ represents a compactification αS of S".

Since $\mathscr{F}$ is a subset of $C^*(S)$, αS need not be βS (generated by $C^*(S)$. Different subsets $\mathscr{F}$ and $\mathscr{G}$ of $C^*(S)$ which separate points and closed sets of S will produce different compactifications of S.

So to each compactification αS we can associate a particular subset, $C_\alpha(S)$, of $C^*(S)$ defined as

$$C_\alpha(S) = \{f|_S \in C^*(S) : f \in C(\alpha S)\}$$

That is, $f \in C_\alpha(S)$ if and only if f extends to $f^\alpha \in C(\alpha S)$.

If αS and γS are two compactifications of S such that $\alpha S \preceq \gamma S$ it is normal to wonder how $C_\alpha(S)$ compares to $C_\gamma(S)$ in $C^*(S)$.

Theorem 21.10 Let αS and γS be two compactifications of S and $C_\alpha(S)$ and $C_\gamma(S)$ be the two subalgebras in $C^*(S)$ corresponding to αS and γS, respectively. Then

$$\alpha S \preceq \gamma S \;\Leftrightarrow\; C_\alpha(S) \subseteq C_\gamma(S)$$

Proof: ($\Rightarrow$) We are given that $\alpha S \preceq \gamma S$. Then there is a continuous function $\pi_{\gamma\to\alpha} : \gamma S \to \alpha S$ such that $\pi_{\gamma\to\alpha}(x) = x$ on S.

Suppose $t \in C_\alpha(S)$. It suffices to show that $t \in C_\gamma(S)$. Then there is a function $t^\alpha : \alpha S \to \mathbb{R}$ such that $t^\alpha|_S = t$. Define the function $g : \gamma S \to \mathbb{R}$ as

$$g = t^\alpha \circ \pi_{\gamma\to\alpha}$$

Since $\pi_{\gamma\to\alpha} : \gamma S \to \alpha S$ and $t^\alpha : \alpha S \to \mathbb{R}$ are both continuous, then g is continuous on γS and

$$g|_S(x) = (t\circ\pi_{\gamma\to\alpha})(x) = t(x)$$

So $t = g|_S \in C_\gamma(S)$. Hence

$$\alpha S \preceq \gamma S \;\;\Rightarrow\;\; C_\alpha(S) \subseteq C_\gamma(S)$$

as required.

($\Leftarrow$) Suppose $C_\alpha(S) \subseteq C_\gamma(S)$. Then every function in $C_\alpha(S)$ extends to γS. Then the function

$$e_{C_\alpha(S)} : S \to T = \prod_{f \in C_\alpha(S)} [a_f, b_f]$$

extends to

$$e^{\alpha}_{C_\alpha(S)}[\alpha S] = \mathrm{cl}_T e_{C_\alpha(S)}[S] = \alpha S$$

Then the function

$$[e^{\gamma}_{C_\alpha(S)}]^{\leftarrow} \circ e^{\alpha}_{C_\alpha(S)}$$

continuously maps γS onto αS where

$$[e_{C_\alpha(S)}]^{\leftarrow} \circ e_{C_\alpha(S)}(x) = x$$

fixing the points of S. Then $\alpha S \preceq \gamma S$.

If $\mathscr{F} \subseteq C^*(S)$, we say that $\mathscr{F}$ is a *subalgebra* of $C^*(S)$ if $\mathscr{F}$ is closed under the usual function operations, addition, subtraction, multiplication and scalar multiplication. If $\mathscr{F}$ is not a subalgebra of $C^*(S)$, then we can enlarge the set $\mathscr{F}$ to a subalgebra, $< \mathscr{F} >$, of $C^*(S)$ which is closed under sums, differences, products and scalar products. (Simply view $< \mathscr{F} >$ as the intersection of all subalgebras of $C^*(S)$ which contain $\mathscr{F}$.) The expression $< \mathscr{F} >$ is to be interpreted as the "subalgebra generated by $\mathscr{F}$".

We define a particular subset, $C_\omega(S)$, of $C^*(S)$ as follows:

$$C_\omega(S) = \{f \in C^*(S) : f^\beta \text{ is constant on } \beta S \backslash S\}$$

The subset $C_\omega(S)$ is clearly a *subalgebra* of $C^*(S)$ which contains all constant functions.

If αS is a compactification of S, then $C(\alpha S)$ is clearly closed under the usual function operations, so the same is true for $C_\alpha(S)$. So $C_\alpha(S)$ is a subalgebra of $C^*(S)$. In fact, if $\alpha S \prec \gamma S$, then $C_\alpha(S)$ is a subalgebra of $C_\gamma(S)$.

Equivalent functions in $C^*(S)$.

In what follows we will refer to pairs of functions in $C^*(S)$ which are "equivalent".

We will say that two functions f and g in $C^*(S)$ are *equivalent functions in* $C^*(S)$ if

$$f - g \in C_\omega(S)$$

The next theorem shows that there are as many compactifications of S as there are subalgebras of $C^*(S)$ of a certain type.

Suppose (S, τ) is a completely regular space and $\mathscr{F}$ is a subset of $C^*(S)$. Suppose the set $\mathscr{G} = \mathscr{F} \cup C_\omega(S)$ in $C^*(S)$ is known to separate points and closed sets. We can invoke Theorem 7.16 to obtain a compactification,

$$\alpha S = \text{cl}_T e_{\mathscr{G}}[S]$$

of S, generated by $\mathscr{G}$. We can associate to the compactification αS, the set $C_\alpha(S) \subseteq C^*(S)$ of real-valued continuous functions. In the following theorem we show conditions that $\mathscr{G}$ must satisfy to guarantee that $<\mathscr{G}> = C_\alpha(S)$.

Theorem 21.11 Let (S, τ) be a completely regular space and $\mathscr{F} \subseteq C^*(S)$. Let $\mathscr{G} = \mathscr{F} \cup C_\omega(S)$. Suppose $\mathscr{G}$ is known to separate points and closed sets of S. Let

$$\alpha S = \text{cl}_T e_{\mathscr{G}}[S]$$

denote the compactification generated by $\mathscr{G}$.
Suppose every function in $C_\alpha(S)$ is equivalent to some function in $\mathscr{F}$. Then $C_\alpha(S) = <\mathscr{G}>$.

Proof: We are given $\mathscr{F}$ is a subset of $C^*(S)$, that $\mathscr{G} = \mathscr{F} \cup C_\omega(S)$ separates points and closed sets of the completely regular space S and

$$\alpha S = \text{cl}_T e_{\mathscr{G}}[S]$$

We are also given that every function in $C_\alpha(S)$ is equivalent to some function in $\mathscr{F}$.

We are required to show that $<\mathscr{G}> = C_\alpha(S)$.

Claim #1: $<\mathscr{G}> \subseteq C_\alpha(S)$.

If $f \in \mathscr{G}$, let I_f be the range of f. Let

$$T = \prod_{f \in \mathscr{G}} \text{cl}_{\mathbb{R}} I_f$$

Since $\mathscr{G}$ separates points and closed sets of S, then the evaluation map, $e : S \to T$ embeds S in T. Let,

$$\alpha S = \text{cl}_T e[S] \subseteq \prod_{f \in \mathscr{G}} \text{cl}_{\mathbb{R}} I_f$$

a compact set which densely contains $e[S]$, a homeomorphic copy of S.

Suppose $g \in \mathscr{G}$.

To show that $g \in C_\alpha(S)$, we are required to show that $g : S \to \mathbb{R}$ extends continuously to some function, $g^\alpha : \alpha S \to \mathbb{R}$.

If π_g is the g^{th}-projection map, then

$$\pi_g : \prod_{f \in \mathscr{G}} \text{cl}_{\mathbb{R}} I_f \to \text{cl}_{\mathbb{R}} I_g$$

where $\alpha S = \text{cl}_T e[S] \subseteq \prod_{f \in \mathscr{G}} \text{cl}_{\mathbb{R}} I_f$ and so,

$$\pi_g|_{\alpha S} : \alpha S \to \text{cl}_{\mathbb{R}} I_g$$

maps αS into $\text{cl}_{\mathbb{R}} I_g$. Let $g^\alpha = \pi_g|_{\alpha S}$. Then $g^\alpha[\alpha S] = g^\alpha[\text{cl}_T e[S]] = \text{cl}_{\mathbb{R}} g[S] \subseteq \text{cl}_{\mathbb{R}} I_g$.

It follows that $g^\alpha : \alpha S \to \text{cl}_{\mathbb{R}} I_g$ and, since $g[S] \subseteq I_g$, for $x \in S$, $g^\alpha|_S(x) = g(x)$.

So g^α is a continuous extension of g from S to αS.

If each $f \in \mathscr{G}$ extends to αS then every f in $< \mathscr{G} >$ extends to αS. So $< \mathscr{G} > \subseteq C_\alpha(S)$. This proves Claim #1.

Claim #2: $C_\alpha(S) \subseteq < \mathscr{G} > = < \mathscr{F} \cup C_\omega(S) >$. Suppose $g \in C_\alpha(S)$. By hypothesis, every g is equivalent to some function in $\mathscr{F}$ so there is a function $f \in \mathscr{F}$ such that $g - f = h \in C_\omega(S)$. Then $g = f + h$. Since both f and h belong to $\mathscr{F} \cup C_\omega(S)$ then $g \in < \mathscr{F} \cup C_\omega(S) >$. We can then conclude that $C_\alpha(S) \subseteq < \mathscr{G} > = < \mathscr{F} \cup C_\omega(S) >$, as claimed.

Then $C_\alpha(S) = < \mathscr{G} > = < \mathscr{F} \cup C_\omega(S) >$, as required.

The reader can expect to encounter, a bit later, similar theorem statements, one of which is called *The Stone-Weierstrass theorem* in 30.3, the other is Theorem 30.4.[7]

[7] The Stone-Weierstrass theorem states: "Let S be a compact topological space. Let $\mathscr{F}$ be a complete subring of $C^*(S)$ which contains the constant functions. If $\mathscr{F}$ separates the points of S, then $\mathscr{F} = C^*(S)$". A consequence of the Stone-Weierstrass statement is the Theorem 30.4. It roughly states that: "If the set, $C^*(S)$ contains a subring, $\mathscr{F}$, which is complete and contains $C_\omega(S)$, then $\mathscr{F} = C_\alpha(S)$ for some compactification, αS, of S."

Suppose γS is a compactification of S and $C_\gamma(S) = \{f|_S \in C(S) : f \in C(\gamma S)\}$. That is, $C_\gamma(S)$ is the set of all function, f, in $C^*(S)$ which extend to $f^\gamma : \gamma S \to \mathbb{R}$. We have shown that $\gamma S \preceq \beta S$ and $C_\gamma(S)$ is a subalgebra of $C^*(S)$. We have also seen that there is a continuous map $\pi_{\beta\to\gamma} : \beta S \to \gamma S$ which fixes the points of S.

In the following lemma we show that we can express the function $\pi_{\beta\to\gamma} : \beta S \to \gamma S$ in a form which better describes the mechanism behind the function itself.

Lemma 21.12 Let γS be a compactification of the space S. Let $\mathscr{G} = C_\gamma(S)$. Then

$$\pi_{\beta\to\gamma} = {e_\mathscr{G}}^{\gamma\leftarrow} \circ {e_\mathscr{G}}^{\beta}$$

where $e_\mathscr{G}$ is the evaluation map generated by $\mathscr{G}$.

Proof: If $f \in C_\gamma(S)$, for $x \in \beta S$, $f^\beta(x) = (f^\gamma \circ \pi_{\beta\to\gamma})(x)$. Then, for $x \in \beta S$,

$$\begin{aligned} {e_\mathscr{G}}^{\beta}(x) &= < f^\beta(x) >_{f\in C_\gamma(S)} \\ &= < (f^\gamma \circ \pi_{\beta\to\gamma})(x) >_{f\in C_\gamma(S)} \\ &= < f^\gamma(\pi_{\beta\to\gamma}(x)) >_{f\in C_\gamma(S)} \\ &= {e_\mathscr{G}}^{\gamma}(\pi_{\beta\to\gamma}(x)) \\ &= ({e_\mathscr{G}}^{\gamma} \circ \pi_{\beta\to\gamma})(x) \end{aligned}$$

Then ${e_\mathscr{G}}^{\beta} = ({e_\mathscr{G}}^{\gamma} \circ \pi_{\beta\to\gamma})$ on βS. Then

$$\pi_{\beta\to\gamma} = {e_\mathscr{G}}^{\gamma\leftarrow} \circ {e_\mathscr{G}}^{\beta}$$

By mimicking the proof of the theorem above we can also prove that if $\gamma S \preceq \alpha S$ then

$$\pi_{\alpha\to\gamma} = {e_\mathscr{G}}^{\gamma\leftarrow} \circ {e_\mathscr{G}}^{\alpha}$$

where $\mathscr{G} = C_\gamma(S)$.

21.7 The one-point compactification of a locally compact space.

Amongst all compactifications of a completely regular space S we can identify two distinguished compactifications: The compactification, βS, and the compactification, ωS, whose outgrowth is a singleton set. We have seen in Theorem 21.6 that the family, $\mathscr{C}$, of all *Hausdorff* compactifications of a completely regular space has a maximal compactification, βS.

In Theorem 18.7, we showed that, given a Hausdorff space (S, τ) and a point $\omega \notin S$, we can construct a larger set,

$$\omega S = S \cup \{\omega\}$$

By first defining, $\mathscr{B}_\omega = \{U \cup \{\omega\} : U \in \tau \text{ and } S \backslash U \text{ is compact }\}$, we showed that,

$$\tau_\omega = \tau \ \cup \ \mathscr{B}_\omega$$

defines a topology on ωS. We then showed that, if S is locally compact, $(\omega S, \tau_\omega)$ is a Hausdorff compact space which densely contains S, hence ωS is a compactification of S with a single element, ω, in its outgrowth.

But in Theorem 18.7, we also showed that, if S is not locally compact, then ωS is *not* Hausdorff and so does not belong to $\mathscr{C}$.

So only locally compact completely regular spaces can have a Hausdorff compactification ωS with a single element in its outgrowth.

For example, since $\mathbb{Q}$ is not locally compact, $\omega\mathbb{Q}$ is not a Hausdorff compactification of $\mathbb{Q}$. Also, we have shown that the Sorgenfrey line $\mathbb{R}_S$ is completely regular and not locally compact. So the compactification $\omega\mathbb{R}_S$ is not Hausdorff.

For future reference, we formally define the one-point compactification.

Definition 21.13 Let (S, τ) be a locally compact Hausdorff non-compact topological space, ω be a point not in S and $\omega S = S \cup \{\omega\}$. If

$$\mathscr{B}_\omega = \{U \cup \{\omega\} : U \in \tau \text{ and } S \backslash U \text{ is compact }\}$$

and $\tau_\omega = \tau \ \cup \ \mathscr{B}_\omega$ then $(\omega S, \tau_\omega)$ is called...

...the one-point compactification of S. [8]

We will, more succinctly, denote the one-point compactification of S by,

$$\omega S = S \cup \{\omega\}$$

---

[8] The one-point compactification of S is also referred to as the *Alexandrov compactification* of S, named after the soviet mathematician, Pavel Alexandrov, (1896-1982).

Note that, if S is locally compact, and αS is any compactification of S, then $\pi_{\alpha\to\omega}$ continuously collapses the outgrowth $\alpha S \backslash S$ down to $\{\omega\}$ and fixes the points of S. So, for any compactification αS,

$$\omega S \preceq \alpha S$$

Since we have chosen the symbol ωS as notation, then, to be consistent with the notation used up to now, we should let

$$C_\omega(S) = \{f|_S : f \in C(\omega S)\}$$

But the symbol, $C_\omega(S)$, has already been used in another sense on page 488 where $C_\omega(S)$ was said to represent

$$\text{“}\{f \in C^*(S) : f^\beta \text{ is constant on } \beta S\backslash S\}\text{”}$$

We verify that these are the same set.

We have,

$$f \in C(\omega S) \quad \Leftrightarrow \quad f|_S^\beta[\beta S\backslash S] = f|_S^\beta[\,\pi^{\leftarrow}_{\beta\to\omega}[\{\omega\}]\,]$$

and since $f^\beta = f^\omega \circ \pi_{\beta\to\omega}$, then $f|_S^\beta[\beta S\backslash S] = f^\omega[\{\omega\}]$

So $f \in C_\omega(S)$ if and only if f^β is constant on $\beta S\backslash S$.

If S is locally compact, ωS is normal, then $C(\omega S)$ separates points and closed sets of S. Hence,

$$C_\omega(S) \textit{ separates points and closed sets of } S.$$

Also note that, for any compactification αS of S,

$$\omega S \preceq \alpha S \Rightarrow C_\omega(S) \subseteq C_\alpha(S)$$

So $C_\omega(S)$ is a subalgebra of $C_\alpha(S)$ for all compactifications, αS.

Amongst all the completely regular spaces, S, the only ones that are open in any compactification are the ones where S is locally compact. We will prove this now.

In the proof of the following theorem we refer to statement (d) of Theorem 18.3. We remind ourselves of what it states: “If S is a Hausdorff space which contains a non-empty subset W, then W is locally compact if and only if W is the intersection of an open subset and a closed subset of S”.

Theorem 21.14 Let S be a completely regular topological space and αS be any compactification of S. Then S is open in αS if and only if S is locally compact.

Proof: We are given that S is completely regular and αS is a compactification of S.

($\Rightarrow$) Suppose S is open in αS. Since the space S is the intersection of the open set S and the closed set αS, by Theorem 18.3, S is locally compact.

($\Leftarrow$) Suppose S is locally compact in αS. By Theorem 18.3 part (d), S is the intersection of an open subset, U, and a closed subset, F. Since $F = \alpha S$ and S is dense in αS, then S is open in αS, as required.

We should also be sure that any two one-point compactifications of a space are equivalent compactifications.

Theorem 21.15 *Uniqueness of* ωS. Let S be a locally compact completely regular topological space. Suppose αS and ωS are both compactifications of S which contain only one point in their compact extension. Then they are equivalent compactifications.

Proof: We are given that S is locally compact completely regular. Suppose $\alpha S = S \cup \{p\}$ where S is embedded in αS, hence is a compactification of S. Suppose $\omega S = S \cup \{\omega\}$ is any other one-point compactification of S. We are required to show that αS and ωS are equivalent compactifications of S.

Consider the map, $h : \alpha S \rightarrow \omega S$, where $h(x) = x$ on S and $h(p) = \omega$. Then h is one-to-one and onto. If U is an open subset of S then $h^{\leftarrow}[U]$ is open in S. If U is an open neighborhood of ω then $\omega S \backslash U$ is a compact subset of S, hence, since $\alpha S \backslash h^{\leftarrow}[U]$ is compact in S, $h^{\leftarrow}[U]$ is an open neighborhood of p. So h is continuous on αS.

It will now suffice to show that h maps open neighborhoods to open neighborhoods. It suffices to show that, if U is an open neighborhood of p, then $h[U]$ is an open neighborhood of ω.

Suppose $p \in U$. Since $\alpha S \backslash U$ is compact and $h|_S$ is continuous, then $h[\alpha S \backslash U]$ is compact. Since h is one-to-one, $h[\alpha S \backslash U] = h[\alpha S] \backslash h[U] = \gamma S \backslash h[U]$, a compact set which doesn't contain ω. So $h[U]$ is open. Hence $h : \alpha S \to \gamma S$ is a homeomorphism. We can conclude that $\alpha S \equiv \gamma S$.

So all compactifications with a single point in the outgrowth are equivalent compactifications.

We now present a variety of examples involving compactifications of a completely regular space.

Example 6. Suppose that S is locally compact and its one-point compactification, ωS, of S is metrizable. Show that S must be second countable.

Solution: Suppose ωS is metrizable. In the proof of theorem 15.8 it is shown that countably compact metric spaces are separable. By Theorem 5.13, a separable metric space is second countable. Since ωS is compact and so is countably compact, then, by combining these two results we obtain that ωS is second countable. By theorem, 5.15, subspaces of second countable spaces are second countable. So S is second countable. As required.[10]

It may happen that βS and ωS are the same compactification. We provide an example where they are not equivalent.

Example 7. Consider the set $S = (0, 1]$ equipped with the usual subspace topology. Determine ωS. Show that the one-point compactification, $\omega S = [0, 1]$, of S is not equivalent to βS.

Solution: Since [0, 1] is a compact set which densely contains S, and the one-point compactification is unique, then $\omega S = [0, 1]$. Note that, since the function $f(x) = \sin \frac{1}{x}$ is a bounded continuous function on S, then it extends to βS. (Plot $f(x) = \sin \frac{1}{x}$.) Verify that f does not extend continuously to $[0, 1]$.

Then $[0, 1]$ cannot be the Stone-Čech compactification of S.

Example 8. Let $S = \mathbb{R}^2$ be equipped with the usual topology. We know that S is locally compact and so has a one-point compactification. If

[10] The converse of the statement in this example is true. That is, "Locally compact second countable spaces have a metrizable one-point compactification" has been proven. But its proof is fairly involved. So we will not show it here.

$\omega S = S \cup \{\omega\}$ represents the one-point compactification of S, a basic open neighborhood of ω in ωS is of the form $\{\omega\} \cup \mathbb{R}^2 \backslash F$, where F is a compact subset of $\mathbb{R}^2$. Since compact subsets of S are closed and bounded we can then more precisely describe such a neighborhood as

$$K = \{\omega\} \ \cup \ \mathbb{R}^2 \backslash \{(x,y) : \|(x,y)\| \leq b\}$$

Example 9. We provide an example which shows that a circle in $\mathbb{R}^2$ is a one-point compactification of the real line, $\mathbb{R}$. Consider the circle S of radius $1/2$ with center, $(0, 1/2)$, in $\mathbb{R}^2$. Suppose we remove from S the point $p = (0, 1)$ at the top of the circle. We refer to such a circle as a "punctured" circle, $S\backslash\{p\}$.

We will show that the subset, $S\backslash\{p\}$ in $\mathbb{R}^2$ is homeomorphic to $\mathbb{R}$. To see this we first visualize a set of lines, L, which pass through the point p, intersecting $S\backslash\{p\}$ and the x-axis. We can view these lines as a function $g : S\backslash\{p\} \to \mathbb{R}$ which links a point (x, y) at $S\backslash\{p\} \cap L$ to a corresponding point $g(x, y)$ at x-axis $\cap L$.

We claim that the function $g : S\backslash\{p\} \to \mathbb{R}$ defined as

$$g(x,y) = \left(\frac{1}{1-y}\right) x$$

is such a function. Each line L, exhibits two right triangles with "slope" $\frac{1-y}{x}$ and $\frac{1}{g(x,y)}$. Since these slopes are equal we obtain the expected function $g(x, y) = \left(\frac{1}{1-y}\right) x$. It is easily verified that g is both one-to-one on $S\backslash\{p\}$ and onto the x-axis. Also, since the point $p = (0, 1)$ does not belong to $S\backslash\{p\}$ and, since g and $g^{\leftarrow}$ respect convergence and limits, then both g and $g^{\leftarrow}$ are continuous. Then the function $g : S\backslash\{p\} \to \mathbb{R}$ maps $S\backslash\{p\}$ homeomorphically onto $\mathbb{R}$. Then the circle S contains a dense copy of $\mathbb{R}$. So S is a one-point compactification representation, $\omega\mathbb{R}$, of $\mathbb{R}$.

Example 10. Using a function which is similar to the one found in the previous example, we can show that a spherical shell in $\mathbb{R}^3$ represents a one point-compactification of $\mathbb{R}^2$. To see this we consider the spherical shell, S of radius $1/2$ with center, $(0, 0, 1/2)$, in $\mathbb{R}^3$. Suppose we remove from S the point $p = (0, 0, 1)$ at the top of the sphere. We obtain a "punctured" sphere, $S\backslash\{p\}$. Consider the function $g : S\backslash\{p\} \to \mathbb{R}^2$ defined as,

$$g(x,y,z) = \left(\frac{1}{1-z}x, \frac{1}{1-z}y\right)$$

Since g is restricted to $S\backslash\{p\}$, $1-z\neq 0$, so g is well-defined. It is a routine exercise to show that $g : S\backslash\{p\}\to\mathbb{R}^2$ is one-to-one on $S\backslash\{p\}$. If $(a,b)\in\mathbb{R}^2$, we have

$$g(a(1-z),b(1-z),z)=\left(\frac{1}{1-z}a(1-z),\frac{1}{1-z}b(1-z)\right)=(a,b)$$

so g is onto $\mathbb{R}^2$. Since both $\pi_1\circ g(x,y,z)=\frac{x}{1-z}$ and $\pi_2\circ g(x,y,z)=\frac{y}{1-z}$ are continuous (when $z\neq 0$), then g is continuous on $S\backslash\{p\}$. Since $g[U\times V\times\{z\}]=\frac{1}{1-z}U\times\frac{1}{1-z}V$ then g is open. We conclude that g maps $S\backslash\{p\}$ homeomorphically onto $\mathbb{R}^2$. The sphere S contains a dense homeomorphic copy of $\mathbb{R}^2$. By definition, S is a one-point compactification representation, $\omega\mathbb{R}^2$, of $\mathbb{R}^2$.

The following theorem generalizes the formula expressed in the two previous examples to n-spheres.

Theorem 21.16 Let S^n denote the n-sphere in $\mathbb{R}^{n+1}$. Then the "punctured n-sphere", S^n minus a single point, $S^n\backslash\{p\}$, is homeomorphic to $\mathbb{R}^n$. Then, S^n is homeomorphic to the one-point compactification, $\omega\mathbb{R}^n$, of $\mathbb{R}^n$.

Proof: We are given that S^n is the n-sphere which is a compact subset of $\mathbb{R}^{n+1}$ and that $p=(0,0,0,\ldots,0,1)$ is a point in $\mathbb{R}^{n+1}$ which belongs to S^n. We will first show that $S^n\backslash\{p\}$ is homeomorphic to $\mathbb{R}^n$. We achieve this by showing that the function $g : S^n\backslash\{p\}\to\mathbb{R}^n$ defined as

$$g(x_1,x_2,\ldots,x_n,x_{n+1})=\frac{1}{1-x_{n+1}}(x_1,x_2,\ldots,x_{n-1},x_n)$$ [12]

maps $S^n\backslash\{p\}$ homeomorphically onto $\mathbb{R}^n$.

Since $p=(0,0,\ldots,1)$, then $1-x_{n+1}\neq 0$, so g is well-defined.

We claim that g is one-to-one on $S^n\backslash\{p\}$ Let $a=(a_1,a_2,\ldots,a_{n+1})$ and $b=(b_1,b_2,\ldots,b_{n+1})$ be distinct points in $S^n\backslash\{p\}$. Then $a_k\neq b_k$ for at least one $k\in\{1,2,\ldots,n+1\}$. Then

$$\frac{1}{1-a_{n+1}}(a_1,a_2,\ldots,a_{n-1},a_n)\neq\frac{1}{1-b_{n+1}}(b_1,b_2,\ldots,b_{n-1},b_n)$$

So g is one-to-one as claimed. The function g is also easily verified to be onto $\mathbb{R}^n$.

[12] The function, g, is known as a *stereographic projection*

To prove continuity of $g : S^n\backslash\{p\} \to \mathbb{R}^n$, it suffices to show that $\pi_i \circ g$ is continuous for each $i \in \{1,\ldots,n\}$, and invoke 7.11. See that

$$\begin{aligned}(\pi_i \circ g)(x_1, x_2, \ldots, x_n, x_{n+1}) &= \pi_i \left[\frac{1}{1-x_{n+1}}(x_1, x_2, \ldots, x_{n-1}, x_n)\right] \\ &= \frac{x_i}{1-x_{n+1}}\end{aligned}$$

so $\pi_i \circ g$ is continuous for each i. So g is continuous on $S^n\backslash\{p\}$.

Also, since π_i is an open map, the function g is an open map.

Then g maps $S^n\backslash\{p\}$ homeomorphically onto $\mathbb{R}^n$, as we claimed earlier. Then the one-point compactification S^n of $S^n\backslash\{p\}$ is equivalent to the one-point compactification of $\mathbb{R}^n$.

The following theorem provides an example of a space, S, such that $\beta S = \omega S$. Note that $\beta S = \omega S$ if and only if every function $f \in C^*(S)$ extends to a function f^β on βS which is constant on $\beta S\backslash S$.

Theorem 21.17 If S is the ordinal space $[0, \omega_1)$ (where ω_1 is the first uncountable ordinal), then $\beta S\backslash S = \{\omega_1\}$ and so $\beta S = \omega S = [0, \omega_1]$, the one-point compactification of S.

Proof: What we are given: The space, S, is the ordinal space $[0, \omega_1)$. Then $\omega S = [0, \omega_1]$ is its one-point Hausdorff compactification. So $S = [0, \omega_1)$ is both a completely regular and locally compact space.

We are required to show that $\beta S = \omega S = [0, \omega_1]$.

In the example on page 351, it is shown that $S = [0, \omega_1)$ is countably compact but non-compact. In Theorem 15.9 it is shown that, if S is countably compact every function in $C(S)$ is bounded and so has a compact image in $\mathbb{R}$.

Let $f \in C^*(S) = C(S)$. Then

$$f^\beta[\beta S] = f^\beta[\mathrm{cl}_{\beta S} S] = \mathrm{cl}_{\mathbb{R}} f[S] = f[S] \subseteq \mathbb{R}$$

To show that $\beta S = \omega S$, it suffices to show that $f^\beta[\beta S\backslash S]$ is a singleton set in $f[S]$ (that is, for any real-valued function f in $C^*(S)$, f^β is constant on $\beta S\backslash S$). Suppose q and q^* are two points in $f^\beta[\beta S\backslash S] \subseteq$

$f[S] \subseteq \mathbb{R}$.

We claim: That $q = q^*$.

Proof of claim: Express $f[S]$ as a net

$$N = \{f(\alpha) : \alpha \in S = [0, \omega_1)\}$$

Open neighborhoods of q and q^* intersect $[0, \omega_1)$. So, for each $n \in \mathbb{N}$, the open balls $B_{1/2n}(q)$ and $B_{1/2n+1}(q^*)$ of radius $1/2n$ and $1/2n+1$ in $\mathbb{R}$ each contain a cofinal subset of the tail end of the net N. Since $f : \beta S \to \mathbb{R}$ is continuous we can then construct a strictly increasing infinite sequence, $\{\alpha_1, \alpha_2, \alpha_3, \ldots\} \subseteq S$ such that $f(\alpha_{2n}) \in B_{1/2n}(q)$ and $f(\alpha_{2n+1}) \in B_{1/2n+1}(q^*)$. Let $\kappa = \sup\{\alpha_n\}$. Since $\{\alpha_n : n = 1, 2, 3, \ldots\}$ is countable $\kappa \neq \omega_1$. Then $\sup\{\alpha_{2n}\} = \kappa = \sup\{\alpha_{2n+1}\}$. Again, since f is continuous on S,

$$q = \lim_{n\to\infty} f(\alpha_{2n}) = f(\kappa) = \lim_{n\to\infty} f(\alpha_{2n+1}) = q^*$$

So $q = q^*$ as claimed.

From this we can conclude that, for all $f \in C^*(S)$, f^β is constant on $\beta S \backslash S$. Then $\beta S = [0, \omega_1] = \omega S$, the one-point compactification of S.

Example 11. Suppose $S = [0, \omega_1)$, where ω_1 is the first uncountable ordinal (equivalently, the set of all countable ordinals). Suppose $f \in C^*(S)$. Show that f is constant on a tail-end of S.

Solution: We are given that f is continuous and real-valued on $S = [0, \omega_1)$ equipped with the ordinal topology. We know that $\beta S = \omega S = [0, \omega_1]$, the one-point compactification of S (Theorem 21.17) and that S is pseudocompact (see page 385). Then $f^\beta(\omega_1)$ is a real number, say q. Hence $\omega_1 \in Z(f^\beta - q)$. Since S is pseudocompact, $Z(f^\beta - q) \cap [0, \omega_1) \neq \varnothing$ (S is pseudocompact implies that $\beta S \backslash S$ cannot contain a zero-set; this is briefly shown in Theorem 21.19). Then $Z(f^\beta - q)$ is cofinal in $[0, \omega_1)$. Let $F = [0, \omega_1) \backslash Z(f^\beta - q)$. We claim that F cannot be cofinal. Suppose F is cofinal. Then there exists a subnet $T = \{g(\alpha_i) : i \in I\}$ of F converging to ω_1. Then T and $Z(f - q)$ are disjoint closed subsets of $[0, \omega_1)$. By the example shown on page 107, T and $Z(f - q)$ cannot be disjoint. So F is not cofinal. Then $Z(f - q)$ is a tail-end of $[0, \omega_1)$. So f is constant on a tail-end of S.

In the following theorem statement an *n-sphere*, S^n, refers to the compact subset of $\mathbb{R}^{n+1}$ defined as $\{\vec{x} \in \mathbb{R}^{n+1} : \|\vec{x}\| = 1\}$. For example, S^2

is a subset of $\mathbb{R}^3$ where $(x, y, z) \in S^2$ if and only if $\|(x, y, z)\| = 1$.

We, of course, know that an infinite discrete space, S, is zero-dimensional (since it has as open base the set of all clopen subsets). But is it necessarily the case that, for any such space, S, ωS is zero-dimensional? One would suspect that it is. In the following example we confirm that ωS is, indeed, a zero-dimensional space.

Example 12. Let S be an infinite discrete space. We consider its one-point compactification, $\omega S = S \cup \{\omega\}$. Show that ωS is a zero-dimensional space.

Solution: We are required to show that ωS has the set of all clopen subsets of ωS as base for open sets. The clopen subsets of ωS are the finite subsets of S and any set, A, containing ω such that $\omega S \setminus A$ is finite

Let $x \in \omega S$ and U be an ωS-open neighborhood of x. To show that ωS is zero-dimensional it suffices to show there is a clopen subset, B, of ωS such that $x \in B \subseteq U$ and invoke part 3 of Theorem 5.3.

Case 1: Suppose $x \in S$. Then $x \in \{x\} \subseteq U$, where $\{x\}$ is clopen in ωS.

Case 2: Suppose $x = \omega$. Then $\omega S \setminus U$ must be finite (by definition of the topology on the one-point compactification). Then U is clopen in ωS.

We conclude that U is a union of clopen subsets of ωS, so ωS is zero-dimensional.

21.8 Topic: Limits of z-ultrafilters in βS.

For what follows, recall the definitions of terms related to z-filters in 14.9.

Suppose $\mathscr{Z} = \{Z(f) : f \in M \subseteq C^*(S)\}$ is a free z-ultrafilter in the locally compact Hausdorff space S, where M is the corresponding free maximal ideal in $C^*(S)$. Let

$$\mathscr{Z}^* = \{\mathrm{cl}_{\beta S} Z(f) : f \in M\}$$

denote a family of closures of the elements in $\mathscr{Z}$. Since βS is compact Hausdorff and $\mathscr{Z}$ is a filter, $\mathscr{Z}^*$ satisfies the finite intersection property. Then $\mathscr{Z}^*$ must have non-empty intersection in βS. Then it is fixed and so must have a unique limit point,

$$\{p\} = \cap\{\mathrm{cl}_{\beta S} Z(f) : f \in M\}$$

in $\beta S \backslash S$. We clearly have $\text{cl}_{\beta S} Z(f) \subseteq Z(f^\beta)$ for each $f \in M \subseteq C^*(S)$. Since $f^\beta|_{Z(f)}$ agrees with f^β on $\text{cl}_{\beta S} Z(f)$, then its extension to $Z(f^\beta)$ agrees with f^β on $Z(f^\beta)$. So

$$\text{cl}_{\beta S} Z(f) = Z(f^\beta) \qquad (^*)$$

So,

> "*... for any free z-ultrafilter,* $\mathscr{Z} = \{Z(f) : f \in M \subseteq C^*(S)\}$ *in* $Z[S]$*, we can write,*
>
> $$\{p\} = \cap\{cl_{\beta S} Z(f) : f \in M\} = \cap\{Z(f^\beta) : p \in Z(f^\beta)\}$$
>
> *where* $p \in \beta S \backslash S$. *So the points in* $\beta S \backslash S$ *are precisely the unique limits of a unique free z-ultrafilter in* $Z[S]$."

Of course, if $\mathscr{Z}$ is a fixed z-ultrafilter in $Z[S]$, then

$$\{p\} = \cap\{Z(f) : f \in M\}$$

for some p in S.

Theorem 21.18 Suppose T is a dense subset of the completely regular space S. Then the following are equivalent:

(a) The subset, T, is C^*-embedded in the space S

(b) Disjoint zero-sets in T have disjoint closures in S.

(c) Every point of S is the limit of a unique z-ultrafilter on T.

Proof: We are given that S is completely regular and T is dense subset of S.

(a) $\Rightarrow$ (b) Suppose T is a C^*-embedded dense subset of S. Let $h, g \in C^*(T)$ be such that $Z(h)$ and $Z(g)$ are disjoint zero-sets of T. Then $Z(h)$ and $Z(g)$ are completely separated in T.[7]

By Urysohn's extension theorem, $Z(h)$ and $Z(g)$ are completely separated in S. Then there exists $t \in C^*(S)$ such that $Z(h) \subseteq Z(t)$ and $Z(g) \subseteq Z(t-1)$. Then $\text{cl}_S Z(h) \subseteq Z(t)$ and $\text{cl}_S Z(g) \subseteq Z(t-1)$. We can then conclude that $\text{cl}_S Z(h) \cap \text{cl}_S Z(g) = \varnothing$, as required.

[7] To see this, note that $Z(|h|+|g|) = \varnothing$ in T and so for the function, $k(x) = \frac{|h(x)|}{[|h(x)|+g(x)|]}$, $Z(h) \subseteq Z(k)$ and $Z(g) \subseteq Z(k-1)$.

(b) ⇒ (c) Suppose that disjoint zero-sets in T have disjoint closures in S, where T is a dense subset of completely regular S. Let Z_1 and Z_2 be disjoint zero-sets in T. We must show that every point in $S \setminus T$ is the limit of a unique z-ultrafilter in T.

Claim #1: We claim that $\text{cl}_S(Z_1 \cap Z_2) = \text{cl}_S Z_1 \cap \text{cl}_S Z_2$.

Proof of claim: Of course, $LHS \subseteq RHS$ is always true. Suppose, on the other hand, that $x \in \text{cl}_S Z_1 \cap \text{cl}_S Z_2$.

We are required to show that $x \in \text{cl}_S(Z_1 \cap Z_2)$. For any zero-set neighborhood, Z, of x, $Z \cap Z_1$ is dense in $Z \cap \text{cl}_S Z_1$ so

$$x \in \text{cl}_S(Z \cap Z_1) \cap \text{cl}_S(Z \cap Z_2) \neq \varnothing \quad (*)$$

Now both $Z \cap Z_1$ and $Z \cap Z_2$ are zero-sets so, by hypothesis, $(Z \cap Z_1) \cap (Z \cap Z_2)$ cannot be empty. So $Z \cap (Z_1 \cap Z_2) \neq \varnothing$. Since Z is any zero-set neighborhood of x, this means that $x \in \text{cl}_S(Z_1 \cap Z_2)$.

We conclude $\text{cl}_S(Z_1 \cap Z_2) = \text{cl}_S Z_1 \cap \text{cl}_S Z_2$, as claimed.

Claim #2: We claim that each point in $S \setminus T$ is the limit of a unique z-ultrafilter on T.

Proof of claim: Let $y \in T$. Then y belongs to the closure of a zero-set, Z, in T. Hence y is the limit point of a z-ultrafilter, $\mathscr{Z}$, in T. Now, if y is also the limit of another z-ultrafilter, $\mathscr{Z}_1$, in T, then $\mathscr{Z}_1$, will contain a zero-set Z_1 which will not intersect some zero-set, Z_2, of $\mathscr{Z}$. Then $y \in \text{cl}_S Z_1 \cap \text{cl}_S Z_2 = \text{cl}_S(Z_1 \cap Z_2) = \varnothing$. We can only conclude that every point in $S \setminus T$ is the limit of a unique z-ultrafilter in T. As claimed.

(c) ⇒ (a) We are given that $S \setminus T$ is a set of limits of free z-ultrafilters on T. We must show that T is C^*-embedded in S. Since $\beta T \setminus T$ is the set of all limit points of free z-ultrafilters on T, we can say that

$$S \setminus T \subseteq \beta T \setminus T$$

Now, since S is dense in βS and T is dense in S, then T is dense in βS so we can view βS as a compactification, say γT, of T with outgrowth

$$\gamma T \setminus T = (\beta S \setminus S) \cup (S \setminus T)$$

Then for the function $\pi_{\beta \to \gamma} : \beta T \to \gamma T$, we have, $\pi_{\beta \to \gamma}[\beta T \setminus T] = \gamma T \setminus T$ where,

$$\pi_{\beta \to \gamma}|_S(x) = x$$

Then, for $f \in C^*(T)$ define $f^* : S \to \mathbb{R}$ as

$$f^*(x) = f^\beta \circ \pi_{\beta \to \gamma}|_S^{\leftarrow}(x)$$

where f^* is seen to be continuous on S and $f^*|_S = f$ on T. So f^* is a continuous extension of f from T to S. We conclude that T is C^*-embedded in S.

Since a completely regular set, S, is C^*-embedded in βS, from part (b) of the above theorem, we can say, for example,

... disjoint zero-sets in S have disjoint closures in βS.

Then, if we partition $\mathbb{N}$ into two infinite subsets A and $B = \mathbb{N}\backslash A$, then A and B are disjoint zero-sets in $\mathbb{N}$ and so have disjoint closures, $\mathrm{cl}_{\mathbb{N}} A$ and $\mathrm{cl}_{\mathbb{N}} B$. Then

$$\beta\mathbb{N} = \mathrm{cl}_{\beta\mathbb{N}}(A \cup B) = \mathrm{cl}_{\beta\mathbb{N}} A \cup \mathrm{cl}_{\beta\mathbb{N}} B$$

the disjoint union of to clopen sets.

Also, recall that $\mathbb{Q}$ is a very particular completely regular dense subset of $\mathbb{R}$, since it is neither open nor locally compact in $\mathbb{R}$. Also recall that $\mathbb{Q}$ is not C^*-embedded in $\mathbb{R}$ (see page 485). From part (b) of the theorem we can then state that $\mathbb{Q}$ must have a pair of disjoint zero sets Z_1 and Z_2 such that $\mathrm{cl}_{\mathbb{R}} Z_1 \cap \mathrm{cl}_{\mathbb{R}} Z_2$ is not empty in $\mathbb{R}$.

21.9 Topic: Pseudocompact spaces revisited.

Recall (from definition 17.11) that a topological space is said to be *pseudocompact* if every continuous real-valued function on S is bounded. That is, if $C(S) = C^*(S)$.

Although the pseudocompact property has a simple and easily understood definition, it turns out that, when not compact, such spaces are not easily recognizable. It will be helpful to obtain a few characterizations. In the following theorem, we show that, for completely regular spaces, pseudocompact spaces are precisely those spaces, S, where the outgrowth, $\beta S\backslash S$, does not contain a zero-set.

Theorem 21.19 A non-compact completely regular space S is pseudocompact if and only if no zero-set Z in $Z[\beta S]$ is entirely contained in $\beta S\backslash S$.

Proof: Let S be a completely regular space.

($\Rightarrow$) Suppose S is a pseudocompact space and $Z(f^\beta) \in Z[\beta S]$. We are required to show that $Z(f^\beta) \cap S \neq \varnothing$.

Suppose not. That is, suppose, $Z(h) \subseteq \beta S \setminus S$ (where $h \in C(\beta S)$). Then, since $h|_S$ is not zero in S, we can define the function $g = 1/h|_S$ and so $g \in C(S)$. Let $z \in Z(h)$. Since z belongs to $\text{cl}_{\beta S} S$, then there is a sequence, $\{x_i\}$ in S, which converges to z. By continuity, the corresponding sequence, $\{h(x_i)\}$ in $\mathbb{R}$, must converge to $h(z) = 0$. So g is unbounded on S, which contradicts the hypothesis which states that S is pseudocompact. So $Z(h)$ must intersect with S, as required.

($\Leftarrow$) Suppose now that, for any $Z \in Z[\beta S]$, $Z \cap S \neq \varnothing$. We are required to show that S is pseudocompact.

Suppose S is not pseudocompact. Then $C(S)$ contains an unbounded function g. Let

$$f = |g| \vee k$$

where $k > 0$. Then f is continuous real-valued unbounded on S. Then, for each $n \in \mathbb{N}$, there exists $x_n \in S$ such that $f(x_n) \in (n, \infty)$. Then $h = 1/f$ is a continuous well-defined real-valued bounded function on S. Since βS is compact, $\{x_n : n \in \mathbb{N}\}$ has a converging subsequence $\{x_{n_i} : i \in \mathbb{N}\}$ with limit, say $x \in \beta S \setminus S$. Since h is continuous and bounded on S, h extends to $h^\beta : \beta S \to \mathbb{R}$ with $h^\beta(x) = 0$. Then $Z(h^\beta) \subseteq \beta S \setminus S$, a contradiction of our hypothesis. Then S is pseudocompact.

It is interesting to note that, in the above theorem, a property of the outgrowth, $\beta S \setminus S$ characterizes a property of the space S.

We point out one easy consequence of the above theorem.

Suppose S is completely regular. If S is pseudocompact and $k = f^\beta(x) \in f^\beta[\beta S \setminus S]$, then there exists $y \in Z(f^\beta - k) \cap S$, so $f^\beta(y) = k \in f[S]$. So $f^\beta[\beta S \setminus S] \subseteq f[S]$.

Conversely, suppose $f^\beta[\beta S \setminus S] \subseteq f[S]$. Suppose $Z(f^\beta - t) \cap \beta S \setminus S \neq \varnothing$. Then there exists $y \in \beta S \setminus S$ such that $f^\beta(y) = t$ and $x \in S$ such that $f(x) = t$. Then $x \in Z(f^\beta - t) \cap S \neq \varnothing$. So S is pseudocompact. We have shown that:

> "*A non-compact completely regular space, S, is pseudocompact if and only if, for every $f \in C^*(S)$, $f^\beta[\beta S \setminus S] \subseteq f[S]$.*"

21.10 Topic: Cardinality of $\beta\mathbb{N}$.

We know the cardinality of the most common sets we encounter (such as $\mathbb{N}$, $\mathbb{Q}$ and $\mathbb{R}$). We can sometimes determine the cardinality of associated sets such as their Stone-Čech compactification. We know the cardinality, $|\mathbb{N}|$, of the set $\mathbb{N}$ is $\aleph_0$, while $|\mathbb{R}| = c = 2^{\aleph_0}$. In the following theorem we compute the cardinality of $\beta\mathbb{N}$.

Theorem 21.20 The cardinality, $|\beta\mathbb{N}|$, of the set $\beta\mathbb{N}$ is 2^c.

Proof: We are given the compactification, $\beta\mathbb{N}$, of $\mathbb{N}$.

Claim #1. That $|\beta\mathbb{N}| \geq 2^c$.

Proof of claim #1: The main idea of the proof of this claim is based on the the fact that there is a function which densely embeds $\mathbb{N}$ into $\prod_{i\in\mathbb{R}}[0,1]_i$, a set whose cardinality is 2^c.

In Theorem 7.10, it is shown that, since $[0,1]$ is separable, then the product space, $K = \prod_{i\in\mathbb{R}}[0,1]_i$, with $|\mathbb{R}| = c$ factors, is also a separable set. This means that K contains a countably infinite dense subset, D. Then there exists a function

$$g : \mathbb{N} \to D \subset K$$

which indexes the elements of D. Note that g is continuous on $\mathbb{N}$ and densely embeds D in K. By Tychonoff's theorem, K is compact. We know that g extends continuously from $\mathbb{N}$ to

$$g^{\beta(K)} : \beta\mathbb{N} \to K$$

Then

$$\begin{aligned} g^{\beta(K)}[\beta\mathbb{N}] &= g^{\beta(K)}[\mathrm{cl}_{\beta\mathbb{N}}\mathbb{N}] \\ &= \mathrm{cl}_K g^{\beta(K)}[\mathbb{N}] \\ &= \mathrm{cl}_K D \\ &= K \end{aligned}$$

Now $|K| = |\prod_{i\in\mathbb{R}}[0,1]| = c^c = 2^c$. (See footnote)[13]. Since $\beta\mathbb{N}$ is the domain of the function $g^{\beta(K)}$ (which could possibly not be one-to-one), then

$$|\beta\mathbb{N}| \geq |K| = 2^c$$

[13]The proof of $|\prod_{i\in\mathbb{R}}[0,1]| = c^c = 2^c$ is shown in an example of *Section* 24.2 of *Set theory: An introduction to Axiomatic Reasoning*, R. André, in which we compute the cardinality of $\mathbb{R}^\mathbb{R}$

as claimed.

Claim #2. That $|\beta\mathbb{N}| \leq |K| = 2^c$.

Proof of claim #2: The main idea of the proof of this claim is based on the fact the the function ${e_{C^*(\mathbb{N})}}^\beta$ embeds $\beta\mathbb{N}$ into $\prod_{i\in I}[a_i, b_i]$, a set whose cardinality is 2^c.

Let $C^*(\mathbb{N}) = \{f_i : i \in I\}$. Then, the cardinality, $|C^*(\mathbb{N})|$ is $|I|$. We know that $C^*(\mathbb{N}) = \mathbb{R}^\mathbb{N}$, so

$$|C^*(\mathbb{N})| = |I| = |\mathbb{R}^\mathbb{N}| = |\mathbb{R}|^{|\mathbb{N}|} = c^{\aleph_0} = c \quad {}_{\text{(See footnote)}}{}^{14}$$

If $f_i \in C^*(\mathbb{N})$ then $f_i[\mathbb{N}] \subseteq [a_i, b_i]$ for some closed interval, $[a_i, b_i]$, in $\mathbb{R}$. The evaluation map, $e_{C^*(\mathbb{N})} : \mathbb{N} \to \prod_{i\in I}[a_i, b_i]$, embeds $\mathbb{N}$ in the product space, $\prod_{i\in I}[a_i, b_i]$. Since $\mathbb{N}$ is C^*-embedded in $\beta\mathbb{N}$, $e_{C^*(\mathbb{N})}$ extends to

$${e_{C^*(\mathbb{N})}}^\beta : \beta\mathbb{N} \to \prod_{i\in I}[a_i, b_i]$$

So

$${e_{C^*(\mathbb{N})}}^\beta[\beta\mathbb{N}] \subseteq \prod_{i\in I}[a_i, b_i] \subseteq \prod_{i\in I}\mathbb{R}_i$$

Then

$$|\beta\mathbb{N}| \;\leq\; \left|\prod_{i\in I}[a_i, b_i]\right| \;\leq\; \left|\prod_{i\in I}\mathbb{R}_i\right| = |\mathbb{R}^I| = |\mathbb{R}|^{|I|} = c^c = 2^c$$

So $|\beta\mathbb{N}| \leq 2^c$, as claimed.

Since $|\beta\mathbb{N}| \leq 2^c$ and $|\beta\mathbb{N}| \geq 2^c$, then $|\beta\mathbb{N}| = 2^c$. We are done.

Now, $\mathbb{N}$ contains countably many points and so

$$|\beta\mathbb{N}\backslash\mathbb{N}| = 2^c$$

Since we can associate to each point in $\beta\mathbb{N}\backslash\mathbb{N}$ a unique free z-ultrafilter in $Z[\mathbb{N}]$, the above theorem confirms that there are 2^c free z-ultrafilters in $Z[\mathbb{N}]$.

[14] The proof of $|\mathbb{R}^\mathbb{N}| = c$ is shown in theorem 25.2 of *Set theory: An introduction to Axiomatic Reasoning*, R. André

21.11 Topic: Compactifying a subset T of $S \subseteq \beta S$.

If T is a non-compact subset of S, it is interesting to reflect on how $\text{cl}_{\beta S}T$ compares with βT. Does it make sense to say that $\beta T \subseteq \beta S$? Under what conditions are $\text{cl}_{\beta S}T$ and βT equivalent compactifications of T? We examine this question in the following example.

Example 13. Let T be a non-empty C^*-embedded non-compact subspace of a completely regular space, S. Show that $\text{cl}_{\beta S}T$ is equivalent to βT.

Solution: We are given that $T \subseteq S$ where T is non-compact and C^*-embedded in S. Since subspaces of completely regular spaces are completely regular, then T is completely regular. Since T is C^*-embedded in S and S is C^*-embedded in βS then T is C^*-embedded in βS. See that $\text{cl}_{\beta S}T$ is a compactification of T. Since T is C^*-embedded in βS it is C^*-embedded in $\text{cl}_{\beta S}T$. So $\text{cl}_{\beta S}T$ is a compact subset of βS which is equivalent to βT.

The converse is easily verified to hold true. We state this formally in the following theorem.

Theorem 21.21 Let T be a non-empty non-compact subspace of the completely regular space S. Then T is C^*-embedded in S if and only if

$$\text{cl}_{\beta S}T \equiv \beta T$$

Proof: The direction ($\Rightarrow$) is proven in the example above.
($\Leftarrow$) Suppose T is such that $\text{cl}_{\beta S}T \equiv \beta T$. We are required to show that T is C^*-embedded in S. Since $\text{cl}_{\beta S}T$ is a compactification equivalent to βT, T is C^*-embedded in $\text{cl}_{\beta S}T$. Since $\text{cl}_{\beta S}T$ is a compact subset of βS it is C^*-embedded in βS. So T is C^*-embedded in S.

Two interesting consequences of the theorem.

1) We previously showed, on page 485, that every closed subset of a metric space is C^*-embedded. As a consequence of the theorem we can then say that,

. . . for any closed , F, of a metric space, S, $cl_{\beta S}F \equiv \beta F$.

2) Another point worth noting relates to a property of $\mathbb{Q}$ as a dense subset of $\mathbb{R}$. We saw on page 485 that $\mathbb{Q}$ is not C^*-embedded in $\mathbb{R}$. Then the theorem allows us to conclude that,

$$\mathrm{cl}_{\beta\mathbb{R}}\mathbb{Q} \not\equiv \beta\mathbb{Q}$$

Since $\mathrm{cl}_{\beta\mathbb{R}}\mathbb{Q}$ is a compactification of $\mathbb{Q}$, in the partial ordering of compactifications of $\mathbb{Q}$

$$\mathrm{cl}_{\beta\mathbb{R}}\mathbb{Q} \prec \beta\mathbb{Q}$$

On the other hand, since $\mathbb{Q}$ is a dense subset of $\beta\mathbb{R}$,

$$\mathrm{cl}_{\beta\mathbb{R}}\mathbb{Q} = \beta\mathbb{R}$$

So $\beta\mathbb{R}$ can be viewed as a compactification of $\mathbb{Q}$. Note that when $\beta\mathbb{R}$ is viewed as a compactification of $\mathbb{Q}$, the outgrowth, $\beta\mathbb{R}\backslash\mathbb{Q}$, of this compactification is dense in $\beta\mathbb{R}$.

Suppose $\alpha\mathbb{Q} = \mathrm{cl}_{\beta\mathbb{R}}\mathbb{Q}$. Then the function $\pi_{\beta\to\alpha} : \beta\mathbb{Q} \to \mathrm{cl}_{\beta\mathbb{R}}\mathbb{Q}$ fixes the points of $\mathbb{Q}$ while mapping $\beta\mathbb{Q}\backslash\mathbb{Q}$ onto $\mathrm{cl}_{\beta\mathbb{R}}\mathbb{Q}\backslash\mathbb{Q} = \beta\mathbb{R}\backslash\mathbb{Q}$.

By Theorem 8.2 part (b) the topology on $\beta\mathbb{R}$ is the quotient topology induced by the quotient map

$$\pi_{\beta\to\alpha} : \beta\mathbb{Q} \to \mathrm{cl}_{\beta\mathbb{R}}\mathbb{Q} = \beta\mathbb{R}$$

(since it maps a compact set onto a compact set, then it is a closed function). This portrays $\beta\mathbb{R}$ as a quotient space induced by $\pi_{\beta\to\alpha}$. While $\beta\mathbb{R}\backslash\mathbb{Q}$ is a quotient space of $\beta\mathbb{Q}\backslash\mathbb{Q}$.[14] It is also clear that $\mathbb{Q}$ is not open in its compactification, $\mathrm{cl}_{\beta\mathbb{R}}\mathbb{Q} = \beta\mathbb{R}$, since every open neighborhood of a rational number will intersect an irrational in its outgrowth. This means that the outgrowth of this compactification of $\mathbb{Q}$ is not compact.

On zero-sets and cozero-sets in $\beta\mathbb{N}$. Since $\mathbb{N}$ is not pseudocompact every zero-set in $\beta\mathbb{N}$ meets $\mathbb{N}$ (by Theorem 21.19). Suppose Z is a proper zero-set in $\mathbb{N}$. Since Z is clopen in $\mathbb{N}$ we can identify Z as the zero-set $Z(g)$ of a characteristic function, g, mapping $\mathbb{N}$ onto $\{0, 1\}$. Since $\mathbb{N}$ is C^*-embedded in $\beta\mathbb{N}$, g extends to a continuous function $g^\beta : \beta\mathbb{N} \to \mathbb{R}$. Then $\mathrm{cl}_{\beta\mathbb{N}}Z(g) \subseteq g^{\beta\leftarrow}(0) = Z(g^\beta)$. Since $\mathbb{N}$ is dense in $\beta\mathbb{N}$, $\mathrm{cl}_{\beta\mathbb{N}}Z(g) = Z(g^\beta)$.

Given that, $g^\beta[\beta\mathbb{N}] = g^\beta[\mathrm{cl}_{\beta\mathbb{N}}\mathbb{N}] \subseteq \mathrm{cl}_{\mathbb{R}}g[\mathbb{N}] = \{0, 1\}$, and $g[\mathbb{N}] = \{0, 1\} \subseteq g^\beta[\beta\mathbb{N}]$, we have that g^β maps $\beta\mathbb{N}$ onto $\{0, 1\}$. Then $Z(g^\beta) \cap$

[14] By definition of "$\preceq$" and "$\prec$" it would be ambiguous (and certainly confusing) to write "$\beta\mathbb{R} \prec \beta\mathbb{Q}$" without some detailed explanation.

$Z(g^\beta - 1) = g^{\leftarrow}(0) \cap g^{\leftarrow}(1) = \varnothing$.

We conclude that $Z(g^\beta)$ and $\text{Cz}(g^\beta)$ are clopen subsets of $\beta\mathbb{N}$.

So $\beta\mathbb{N}$ is zero-dimensional. It is locally compact so it is totally disconnected.

Example 14. The compactification, $\beta\mathbb{N}$, is separable (since $\mathbb{N}$ is a dense subset of $\beta\mathbb{N}$). Show that $\beta\mathbb{N}\backslash\mathbb{N}$ is *not* separable.

Solution: Suppose that $\beta\mathbb{N}\backslash\mathbb{N}$ is separable. Then, $\beta\mathbb{N}\backslash\mathbb{N}$ contains a dense countable subset

$$D = \{x_i : i \in \mathbb{N}\}$$

Our strategy is to prove that D cannot be dense $\beta\mathbb{N}\backslash\mathbb{N}$. Since the cardinality of $\beta\mathbb{N}$ is 2^c (shown above), we can fix two points

$$n^* \in (\beta\mathbb{N}\backslash\mathbb{N})\backslash D \text{ and } n \in \mathbb{N}$$

Now, for each $i \in \mathbb{N}$, $\{x_i\} \subseteq D$ and $\{n^*, n\} \subseteq \beta\mathbb{N}\backslash D$ form disjoint closed subsets of $\beta\mathbb{N}$.

Since $\beta\mathbb{N}$ is normal, there is a clopen zero-set, $Z_i = Z(g^\beta)$ such that

$$x_i \in Z_i \quad \text{and} \quad \{n^*, n\} \subseteq \beta\mathbb{N}\backslash Z_i$$

Since $\beta\mathbb{N}$ was shown above to be zero-dimensional, then

$$\{\beta\mathbb{N}\backslash Z_i : i \in \mathbb{N}\}$$

is a countably infinite family of zero-set clopen neighborhoods of $\{n^*, n\}$.

If $W = \cap\{\beta\mathbb{N}\backslash Z_i : i \in \mathbb{N}\}$, then W is a countable intersection of zero-sets and so is, itself, a zero-set which contains $\{n^*, n\}$ (see page 235) which does not intersect D. Since $n^* \in \beta\mathbb{N}\backslash\mathbb{N} \cap W$, then $W \cap \beta\mathbb{N}\backslash\mathbb{N}$ is a non-empty clopen subset of $\beta\mathbb{N}\backslash\mathbb{N}$ which does not intersect D. Since D is dense in $\beta\mathbb{N}\backslash\mathbb{N}$ we have a contradiction.

We conclude that $\beta\mathbb{N}\backslash\mathbb{N}$ is *not* separable.

Since subspaces of separable metrizable spaces were shown to be separable (see page 102), then,

> $\ldots\beta\mathbb{N}$ *is not a metrizable space.*

In what follows, the cardinality of $\beta\mathbb{N}$ will help us determine the cardinalities of $\beta\mathbb{R}$ and $\beta\mathbb{Q}$.

Theorem 21.22 Let A be an infinite subset of $\beta\mathbb{N}$. Then the cardinality of $\text{cl}_{\beta\mathbb{N}}A$ is 2^c.

Proof: We are given that A be an infinite subset of $\beta\mathbb{N}$. We are required to show that $|\text{cl}_{\beta\mathbb{N}}A| = 2^c$.

Since $\text{cl}_{\beta\mathbb{N}}A$ is an infinite Hausdorff space we can inductively construct a countably infinite copy, $K = \{n_i : i \in \mathbb{N}\}$, of $\mathbb{N}$ in $\text{cl}_{\beta\mathbb{N}}A$.

Consider the countable completely regular subset, $D = \mathbb{N} \cup K$, of $\beta\mathbb{N}$. Then since $\mathbb{N} \subseteq D$, $\text{cl}_{\beta\mathbb{N}}D = \beta\mathbb{N}$. So, by Theorem 21.21 above, D is C^*-embedded in $\beta\mathbb{N}$.

We claim that K is C^*-embedded in D. To see this first note that for every $n \in \mathbb{N}$, $\{n\}$ is open in D and so K is closed in D. Also, every open cover of D has a countable subcover and so D is Lindelöf. Then the subspace D is normal (See Theorem 16.6). Then, by Theorem 10.9, K is C^*-embedded in D, as claimed.

Since K is C^*-embedded in D and D is C^*-embedded in $\beta\mathbb{N}$, then K is C^*-embedded in $\beta\mathbb{N}$. Then $\text{cl}_{\beta\mathbb{N}}K = \beta K \subseteq \text{cl}_{\beta\mathbb{N}}A$. Since the cardinality of βK has been shown to be 2^c, the cardinality of $\text{cl}_{\beta\mathbb{N}}A$ is 2^c. As required.

Corollary 21.23 If $p \in \beta\mathbb{N}\backslash\mathbb{N}$, there is no infinite sequence of distinct points in $\beta\mathbb{N}$ which converges to p.

Proof: Suppose $p \in \beta\mathbb{N}$. Suppose $A = \{x_n : n \in \mathbb{N}\}$ is a sequence in $\beta\mathbb{N}$ which converges to p. If the elements of A are distinct then $\text{cl}_{\beta\mathbb{N}}A = A \cup \{p\}$ is an infinite closed subspace of $\beta\mathbb{N}$ of cardinality $\aleph_0$. This contradicts the previous theorem stating that the cardinality of $\text{cl}_{\beta\mathbb{N}}A$ is 2^c. Then no infinite sequence of distinct points in $\beta\mathbb{N}$ can converge to p.

This allows us to exhibit a compact space which is not sequentially compact.

Corollary 21.24 The compactification $\beta\mathbb{N}$ is not a sequentially compact space.

Proof: Suppose $A = \{x_n : n \in \mathbb{N}\}$ is a sequence of distinct points in $\beta\mathbb{N}$. If $\beta\mathbb{N}$ is a sequentially compact space then the sequence A must have a subsequence $A_1 = \{x_{n_j} : j \in \mathbb{N}\}$ of distinct points which converges to some point p in $\beta\mathbb{N}$. Then $\text{cl}_{\beta\mathbb{N}} A_1 = A_1 \cup \{p\}$ has cardinality $\aleph_0$ contradicting the statement in Corollary 21.22. So $\beta\mathbb{N}$ is not sequentially compact.

Theorem 21.25 The sets $\beta\mathbb{R}$ and $\beta\mathbb{Q}$ each have a cardinality equal to 2^c.

Proof: We have previously shown that $|\beta\mathbb{N}| = 2^c$.

Now $\mathbb{N}$ is a closed subset of $\mathbb{Q}$ and so is C^*-embedded in $\mathbb{Q}$ (by Theorem 10.9). Since $\mathbb{Q}$ is C^*-embedded in $\beta\mathbb{Q}$, then $\mathbb{N}$ is C^*-embedded in $\beta\mathbb{Q}$. Then, by Theorem 21.21,

$$\text{cl}_{\beta\mathbb{Q}}\mathbb{N} \equiv \beta\mathbb{N} \subseteq \beta\mathbb{Q}$$

So $|\beta\mathbb{Q}| \geq |\beta\mathbb{N}| = 2^c$.

To show that $|\beta\mathbb{Q}| \leq |\beta\mathbb{N}|$ it now suffices to produce a function which maps $\beta\mathbb{N}$ onto $\beta\mathbb{Q}$. Let $q : \mathbb{N} \to \mathbb{Q}$ be any continuous function which maps $\mathbb{N}$ onto $\mathbb{Q}$, viewed as a subset of the space $\beta\mathbb{Q}$. Then $q : \mathbb{N} \to \beta\mathbb{Q}$ continuously maps $\mathbb{N}$ into $\beta\mathbb{Q}$. Then q extends to the continuous function $q^{\beta(\beta\mathbb{Q})} : \beta\mathbb{N} \to \beta\mathbb{Q}$ (See Theorem 21.5). Then

$$\begin{aligned} q^{\beta(\beta\mathbb{Q})}[\beta\mathbb{N}] &= q^{\beta(\beta\mathbb{Q})}[\text{cl}_{\beta\mathbb{N}}\mathbb{N}] \\ &= \text{cl}_{\beta\mathbb{Q}} q[\mathbb{N}] \\ &= \text{cl}_{\beta\mathbb{Q}}\mathbb{Q} \\ &= \beta\mathbb{Q} \end{aligned}$$

Then $|\beta\mathbb{Q}| \leq |\beta\mathbb{N}| = 2^c$.

We conclude that

$$|\beta\mathbb{Q}| = 2^c$$

Similarly, $\mathbb{N}$ is a closed subset of $\mathbb{R}$ and so is C^*-embedded in $\mathbb{R}$. Since $\mathbb{R}$ is C^*-embedded in $\beta\mathbb{R}$, then $\mathbb{N}$ is C^*-embedded in $\beta\mathbb{R}$. Then $\text{cl}_{\beta\mathbb{R}}\mathbb{N} \equiv \beta\mathbb{N} \subseteq \beta\mathbb{R}$ (Theorem 21.21). So $|\beta\mathbb{R}| \geq |\beta\mathbb{N}|$. Then

$$|\beta\mathbb{R}| \geq 2^c$$

Let $q : \mathbb{Q} \to \mathbb{Q}$ be the inclusion function which maps $\mathbb{Q}$ onto $\mathbb{Q}$, viewed as a subset of the space $\beta\mathbb{R}$. Hence $q : \mathbb{Q} \to \beta\mathbb{R}$. Then q extends to the continuous function $q^{\beta(\beta\mathbb{R})} : \beta\mathbb{Q} \to \beta\mathbb{R}$ (Theorem 21.5). Then

$$\begin{aligned} q^{\beta(\beta\mathbb{R})}[\beta\mathbb{Q}] &= q^{\beta(\beta\mathbb{R})}[\mathrm{cl}_{\beta\mathbb{Q}}\mathbb{Q}] \\ &= \mathrm{cl}_{\beta\mathbb{R}}q[\mathbb{Q}] \\ &= \mathrm{cl}_{\beta\mathbb{R}}\mathbb{Q} \\ &= \beta\mathbb{R} \end{aligned}$$

Then $|\beta\mathbb{R}| \leq |\beta\mathbb{Q}|$. So

$$|\beta\mathbb{R}| \leq 2^c$$

We conclude that

$$|\beta\mathbb{R}| = 2^c$$

Remarks. 1) There are arguments in the proof of Theorem 22.16 that also support his result. Sets which have the same cardinality are said be *equipotent sets*. So $\beta\mathbb{N}$, $\beta\mathbb{Q}$ and $\beta\mathbb{R}$ are equipotent sets.

2) Since $\mathbb{N}$ is C^*-embedded in $\mathbb{R}$, then $\mathrm{cl}_{\beta\mathbb{R}}\mathbb{N} = \beta\mathbb{N} \subseteq \beta\mathbb{R}$. Also, $\mathrm{cl}_{\beta\mathbb{R}}\mathbb{N}\backslash\mathbb{N} \equiv \beta\mathbb{N}\backslash\mathbb{N} \subseteq \beta\mathbb{R}\backslash\mathbb{R}$. Since $|\beta\mathbb{N}\backslash\mathbb{N}| = 2^c$, $|\beta\mathbb{R}\backslash\mathbb{R}| = 2^c$.

> *Then there can be no real-valued function that is one-to-one on* $\beta\mathbb{R}\backslash\mathbb{R}$.

The status of $\beta\mathbb{Q}\backslash\mathbb{Q}$ *as a subset of* $\beta\mathbb{Q}$. Note that, when discussing the Stone-Cěch compactification of a completely regular space, $\beta\mathbb{Q}$ exhibits some peculiarities (mostly since $\mathbb{Q}$ is not locally compact and so is not open in $\beta\mathbb{Q}$). For example, suppose U is a $\beta\mathbb{Q}$-open subset entirely contained in $\mathbb{Q}$. Then $\mathrm{cl}_{\beta\mathbb{Q}}U$ is compact. Suppose $y \in U \cap \mathbb{Q}$. Then there is a $\beta\mathbb{Q}$-open neighborhood V of y such that

$$y \in V \subseteq \mathrm{cl}_{\beta\mathbb{Q}}V \subseteq U \subseteq \mathbb{Q}$$

Then $\mathrm{cl}_{\mathbb{Q}}V$ is a compact neighborhood of y in $\mathbb{Q}$. But we know that points in $\mathbb{Q}$ do not have compact neighborhoods. Then there can be no $\beta\mathbb{Q}$-open subset entirely contained in $\mathbb{Q}$. That is, every $\beta\mathbb{Q}$-open subset intersects $\beta\mathbb{Q}\backslash\mathbb{Q}$. We conclude that,

> *. . . the subset,* $\beta\mathbb{Q}\backslash\mathbb{Q}$*, is dense in* $\beta\mathbb{Q}$.

More on $\beta\mathbb{Q}$.

Example 15. Show that $\beta\mathbb{Q}$ is totally disconnected.

Solution: We know that $\mathbb{Q}$ is totally disconnected (see page 438).

Suppose p and q are distinct points in $\beta\mathbb{Q}$. Since $\beta\mathbb{Q}$ is normal, then there exists disjoint compact $\beta\mathbb{Q}$-neighborhoods U and V containing p and q, respectively. Then $U \cap \mathbb{Q}$ and $V \cap \mathbb{Q}$ are disjoint closed subsets of $\mathbb{Q}$. By Corollary 20.26, there exists a clopen subset A of $\mathbb{Q}$ which separates $U \cap \mathbb{Q}$ and $V \cap \mathbb{Q}$. The set A is a zero-set, say $Z(g)$ for a characteristic function $g : \mathbb{Q} \to \{0, 1\}$ in $\mathbb{Q}$; then $\text{cl}_{\beta\mathbb{Q}} A = Z(g^\beta)$ and $\text{cl}_{\beta\mathbb{Q}} \mathbb{Q} \setminus A = Z(g^\beta - 1)$. Then $\text{cl}_{\beta\mathbb{Q}} A$ is a clopen subset of $\beta\mathbb{Q}$ which separates p and q. So $\beta\mathbb{Q}$ is totally disconnected.

Example 16. Show that if $\mathbb{R}_S$ denotes the Sorgenfrey line, $\beta\mathbb{R}_S$ is totally disconnected.

Solution: We know that $\mathbb{R}_S$ is totally disconnected (see page 438). Suppose p and q are distinct points in $\beta\mathbb{R}_S$. Then there exists disjoint compact neighborhoods U and V containing p and q, respectively. Then $U \cap \mathbb{R}_S$ and $V \cap \mathbb{R}_S$ are disjoint closed subsets of $\mathbb{R}_S$. By Corollary 20.26, there exists a clopen subset A which separates $U \cap \mathbb{Q}$ and $V \cap \mathbb{Q}$. Then $\text{cl}_{\beta\mathbb{Q}} A$ is a clopen subset of $\beta\mathbb{R}_S$ which separates p and q.

21.12 Compact spaces as a β-outgrowth, $\beta S \setminus S$, for some S.

If we are given a compact Hausdorff space, T, we may wonder whether it is the outgrowth of some topological space. We can answer this question in the affirmative. We can show that all such spaces will appear as the outgrowth, $\beta S \setminus S$, of at least one space S. We will first prove this statement for the particular case where T is a countably infinite compact set. The proof for the more general case where there is no restriction on the cardinality of T will immediately follow. The general proof involves larger cardinals but adopts the same strategy as the one where T is countably infinite.

Theorem 21.26 Every countably infinite compact Hausdorff topological space, T, is the β-outgrowth of some non-compact pseudocompact Hausdorff topological space, S. That is, there is a topological space S, such that βS has an outgrowth $\beta S \setminus S$ which is homeomorphic to T.

Proof: Let T be a countably infinite compact Hausdorff space. The cardinality of T is the first countably infinite ordinal, ω_0, while $\omega_1 = [0, \omega_1)$ represents the first uncountable limit ordinal. Equivalently, it is the least ordinal of cardinality greater than ω_0. Let

$$S = [0, \omega_1) \times T$$

We know that $[0, \omega_1] = \omega_1 + 1$ is compact (see example on page 351). Since ω_1 is a limit ordinal, $[0, \omega_1)$ is dense in $[0, \omega_1]$ (see example on page 106). Since both T and $[0, \omega_1] = \omega_1 + 1$ are compact, then so is the product $[0, \omega_1] \times T$. Also,

$$S = \omega_1 \times T = [0, \omega_1) \times T \subseteq_{\text{dense}} (\omega_1 + 1) \times T = [0, \omega_1] \times T$$

Then $[0, \omega_1] \times T$ is a compactification, say αS, of $S = \omega_1 \times T$ where the outgrowth, $\alpha S \backslash S = \{\omega_1\} \times T$, is a homeomorphic copy of T.

To prove the theorem it will suffice to show that αS is a compactification which is equivalent to βS. To do this, we will show that S is C^*-embedded in $[0, \omega_1] \times T$.

For each $p \in T$, let

$$Y_p = [0, \omega_1) \times \{p\}$$

Then, Y_p is pseudocompact (since in example 1, on page 351, we see that $[0, \omega_1)$ is countably compact and by Theorem 15.9 the continuous real-valued image of Y_p is compact in $\mathbb{R}$). By Theorem 21.17,

$$\beta Y_p = \omega Y_p = Y_p \cup \{\kappa_p\}$$

where $\beta Y_p \backslash Y_p$ is a singleton set, $\{\kappa_p\}$. So Y_p is a copy of $\omega_1 = [0, \omega_1)$ and $Y_p \cup \{\kappa_p\}$ is a copy of $\omega_1 + 1 = [0, \omega_1]$.

Since Y_p is C^*-embedded in $Y_p \cup \{\kappa_p\}$, for $f \in C^*(Y_p)$, f extends continuously to $f^\beta : \beta Y_p \to \mathbb{R}$ from which we obtain a unique real number value for $f^\beta(\kappa_p)$.

In the example found on page 498, it is shown that f must be constant on a tail-end, say $[\gamma_p, \omega_1) \times \{p\}$, of Y_p. So for this choice of $p \in T$, f^β is constant $[\gamma_p, \omega_1) \times \{p\}$

We now consider a function $f \in C^*(S)$ (rather than $f \in C^*(Y_p)$). As p ranges through T, the set $\{\gamma_p : p \in T\}$ contains at most countably infinite ordinals and so is not cofinal in $[0, \omega_1)$. Then $\sup\{\gamma_p : p \in T\} = \delta < \omega_1$. So for $f \in C^*(S)$, for each $p \in T$, $f^\beta|_{Y_p}$ is constant on $\{\,[\,\delta, \omega_1) \times \{p\}\,\}$. So $f : S \to \mathbb{R}$ is constant on $[\delta, \omega_1) \times T$. So, for each p, we define, $f^\beta(\omega_1, p) = \kappa_p$. Then $f^\beta : \beta S \to \mathbb{R}$ can easily be verified to be continuous on βS. We can conclude that every function f in $C^*(S)$ extends to $f^\alpha \in C(\alpha S)$. So S is C^*-embedded in $[0, \omega_1] \times T$. This implies $\beta S = [0, \omega_1] \times T$. Hence $\{\omega_1\} \times T$ is the outgrowth $\beta S \backslash S$.

We now verify that S is a pseudocompact space. Consider a point (ω_1, p) in $\beta S \backslash S = \{\omega_1\} \times T$. Suppose $Z \in Z[\beta S]$ which contains (ω_1, p). Then $Z \cap \{\omega_1\} \times T$ is a zero-set in $Z[[0, \omega_1) \times \{p\} \cup \{(\omega_1, p)\}]$. Since $[0, \omega_1) \times \{p\}$ is pseudocompact $Z \cap \{\omega_1\} \times T$ must intersect $[0, \omega_1) \times \{p\}$. So Z must intersect $S = [0, \omega_1) \times T$. Then S is pseudocompact.

Theorem 21.27 Every compact Hausdorff topological space, T, is the β-outgrowth of some locally compact non-compact Hausdorff topological space, S. That is, there is a topological space S, such that βS has an outgrowth $\beta S \backslash S$ which is homeomorphic to T.

Proof : The proof mimics the one provided for the above theorem but with the added complexity of having to deal with the notion of "cofinality" of larger ordinal numbers.[16]

We are given a compact Hausdorff space T. Suppose the cardinality of T is κ. Let ω_n be a initial ordinal where n is a natural number and $\omega_n > \kappa$. The cofinality, $\text{cf}(\omega_n)$, of ω_n is ω_n, so no set of *ordinality* less than ω_n is cofinal in ω_n.

We know that $[0, \omega_n] = \omega_n + 1$ is compact (see example on page 351). Since ω_n is a limit ordinal, then $[0, \omega_n)$ is dense in $[0, \omega_n]$ (see example on page 106). Since both T and $[0, \omega_n] = \omega_1 + 1$ are compact, then so is the product $[0, \omega_n] \times T$.

Also,

$$S = \omega_n \times T = [0, \omega_n) \times T \subseteq_{\text{dense}} (\omega_n + 1) \times T = [0, \omega_n] \times T$$

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[16] The cofinality of an ordinal α, denoted as $\text{cf}(\alpha)$, is the smallest ordinal γ that is the ordinality of a cofinal subset of α. The set A is cofinal in α if for every ordinal γ, $A \cap [\gamma, \alpha) \neq \varnothing$. Equivalently, $\sup A = \alpha$. The cofinality of a partially ordered set A can also be defined as the least ordinal γ such that there is a function from γ to A with cofinal image. For example, the cofinality, $\text{cf}(\omega_1)$, is ω_1 since the only ordinal which is cofinal in ω_1 is ω_1 itself. Every ordinal less than ω_1 is countable. Any ordinal α such that $\text{cf}(\alpha) = \alpha$ is called a

regular ordinal

A cardinal κ is regular if its cofinality is equal to κ. The cofinality of a cardinal κ, denoted $\text{cf}(\kappa)$, is the smallest cardinality of a set of ordinals less than κ whose supremum is κ. Relevant to the proof of this theorem is the fact that $\{\omega_n : n \in \mathbb{N}\}$ is a set of regular ordinals. That is, $\text{cf}(\omega_n) = \omega_n$. Equivalently, $\{\aleph_n : n \in \mathbb{N}\}$ is a set of regular cardinals.

Then $[0, \omega_n] \times T$ is a compactification, say αS, of $S = \omega_n \times T$ where the outgrowth, $\alpha S \backslash S = \{\omega_n\} \times T$, is a homeomorphic copy of T.

We claim that αS is equivalent to βS.

For each $p \in T$, let

$$Y_p = [0, \omega_n) \times \{p\}$$

Then, Y_p is pseudocompact (just as in example 1, on page 351, we see that $[0, \omega_n)$ is countably compact and by Theorem 15.9 the continuous real-valued image of Y_p is compact in $\mathbb{R}$). By Theorem 21.17,

$$\beta Y_p = \omega Y_p = Y_p \cup \{\kappa_p\}$$

where $\beta Y_p \backslash Y_p$ is a singleton set, $\{\kappa_p\}$. So Y_p is a copy of $\omega_n = [0, \omega_n)$ and $Y_p \cup \{\kappa_p\}$ is a copy of $\omega_n + 1 = [0, \omega_n]$.

Since Y_p is C^*-embedded in $Y_p \cup \{\kappa_p\}$, for $f \in C^*(Y_p)$, f extends continuously to $f^\beta : \beta Y_p \to \mathbb{R}$ so f^β takes on a unique real number value κ_p.

Arguing as in the example found on page 498, f must be constant on a tail-end, say $[\gamma_p, \omega_n) \times \{p\}$, of Y_p. So for this choice of $p \in T$, f^β is constant $[\gamma_p, \omega_n) \times \{p\}$

We now consider a function $f \in C^*(S)$ (rather than $f \in C^*(Y_p)$). As p ranges through T, the cardinality of the set, $\{\gamma_p : p \in T\}$, is κ, and so $\{\gamma_p : p \in T\}$ is not cofinal in $[0, \omega_n)$. Then $\sup\{\gamma_p : p \in T\} = \delta < \omega_n$. So for $f \in C^*(S)$, for each $p \in T$, $f^\beta|_{Y_p}$ is constant on $\{ [\, \delta, \omega_n) \times \{p\} \}$. So $f : S \to \mathbb{R}$ is constant on $[\delta, \omega_n) \times T$. So, for each p, we define, $f^\beta(\omega_n, p) = f^\beta|_{Y_p}(\kappa_p)$. Then $f^\beta : \beta S \to \mathbb{R}$ can easily be verified to be continuous on βS. We can conclude that every function f in $C^*(S)$ extends to $f^\alpha \in C(\alpha S)$. So S is C^*-embedded in $[0, \omega_n] \times T$. This implies $\beta S = [0, \omega_n] \times T$. Hence $\{\omega_n\} \times T$ is the outgrowth $\beta S \backslash S$.

As in the previous theorem, S is a pseudocompact space.

21.13 Topic: Extremally disconnected spaces revisited.

Now that we have explored how a completely regular space, S, relates to a compact set T, in which it is both dense and C^*-embedded, we are able to establish a few extra characterizations of those somewhat mystifying spaces previously referred to as "extremally disconnected".

Recall that extremally disconnected spaces are those spaces whose open sets have open closures.

Theorem 21.28 For a given completely regular topological space (S, τ), the following four statements are equivalent:

(a) The space (S, τ) is extremally disconnected.

(b) Each dense subspace of S is C^*-embedded in S.

(c) Each open subspace of S is C^*-embedded in S.

(d) The space βS is extremally disconnected.

Proof: We are given a completely regular topological space S,

(a) $\Rightarrow$ (b) Suppose S is extremally disconnected and let D be a dense subset of S. We will set up the proof so as to invoke the Urysohn's extension theorem. Let F and K be completely separated sets in D. Then there exists disjoint zero-sets $Z_1 = Z(f)$ and $Z_2 = Z(g)$ containing F and K, respectively. Let $h(x) = |f(x)|/(|f(x)| + |g(x)|)$. Then $h[Z(f)] = \{0\}$ and $h[Z(g)] = \{1\}$. Since $0 \vee (h \wedge 1)$ is continuous on $[0, 1]$, $A = h^{\leftarrow}[[0, 1/3)]$ and $B = h^{\leftarrow}[(2/3, 1]]$ form disjoint D-open subsets which separate Z_1 and Z_2, respectively. Let $A^*, B^* \in \tau$ such that $A = A^* \cap D$ and $B = B^* \cap D$. Since D is dense in S then $A^* \cap B^* = \varnothing$. Also, since S is extremally disconnected, $\mathrm{cl}_S A^*$ and $\mathrm{cl}_S B^*$ are clopen subsets of S. Then a characteristic function on $\mathrm{cl}_S A^*$ separates Z_1 and Z_2. By Urysohn' extension theorem, D is C^*-embedded in S.

(b) $\Rightarrow$ (c) Suppose each dense subset of S is a C^*-embedded subset of S. Let U be an open subset of S. We are required to show that U is C^*-embedded in S. Let $D = U \cup S \backslash \mathrm{cl}_S U$. Then U is clopen in D. Then U is C^*-embedded in D. Since, by hypothesis, D is C^*-embedded, U is C^*-embedded, in S, as required.

(c) $\Rightarrow$ (d) Suppose each open subset of S is a C^*-embedded subset of S. We are required to show that βS is extremally disconnected. Let U be an open subset of βS. Let U^* denote the corresponding open subset, $U \cap S$, of S. Consider the open subset,

$$T = U^* \cup S \backslash \mathrm{cl}_S U^*$$

of S. The set T is the union of two open subsets of S, so U^* and $S \backslash \mathrm{cl}_S U^*$ partition T into two clopen subsets of T. Then the characteristic function $f = \chi_{U^*}$ belongs to $C^*(T)$. By hypothesis, the function f extends to $f^* \in C^*(S)$, which, in turn, extends to $f^\beta \in C(\beta S)$. We then have,

$$\begin{aligned} f^\beta[\mathrm{cl}_{\beta S} U] &= f^\beta[\mathrm{cl}_{\beta S}(U \cap S)] \\ &= \mathrm{cl}_{\beta S} f^*[U \cap S] \\ &= \mathrm{cl}_{\beta S} \chi_{U^*}[U^*] \\ &= \{1\} \end{aligned}$$

For the same reasons, $f^\beta[\mathrm{cl}_{\beta S}[S \backslash \mathrm{cl}_{\beta S} U] = \{0\}$.
Then

$$\mathrm{cl}_{\beta S} U \cap \mathrm{cl}_{\beta S}(S \backslash \mathrm{cl}_{\beta S} U) = \varnothing$$

Since

$$\mathrm{cl}_{\beta S} U \cup \mathrm{cl}_{\beta S}(S \backslash \mathrm{cl}_{\beta S} U) = \beta S$$

then $\mathrm{cl}_{\beta S} U$ is clopen.

(d) $\Rightarrow$ (a) Follows from part (b) of Theorem 20.28.

Example 17. Let the function $f : \mathbb{R} \to \omega\mathbb{R}^+$ be defined as

$$f(x) = x^2 |\sin(x)|$$

where $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$ and $\omega\mathbb{R}^+ = \mathbb{R}^+ \cup \{\infty\}$, the one-point compactification of $\mathbb{R}^+$. Observe that f maps $\mathbb{R}$ onto $\mathbb{R}^+$. Since $\mathbb{R}^+ \cup \{\infty\}$ is compact the function $f : \mathbb{R} \to \omega\mathbb{R}^+$ extends continuously to $f^{\beta(\omega)} : \beta\mathbb{R} \to \omega\mathbb{R}^+$. Describe $\beta\mathbb{R} \backslash \mathbb{R}$ in terms of the domain of $f^{\beta(\omega)}$.

Solution: Note that,

$$\begin{aligned} f^{\beta(\omega)}[\beta\mathbb{R}] &= f^{\beta(\omega)}[\mathrm{cl}_{\beta\mathbb{R}}[\mathbb{R}]] \\ &= \mathrm{cl}_{\omega\mathbb{R}^+} f[\mathbb{R}] \\ &= \mathrm{cl}_{\omega\mathbb{R}^+} \mathbb{R}^+ \\ &= \mathbb{R}^+ \cup \{\infty\} \end{aligned}$$

Note that for each $x \in \mathbb{R}^+$, $f^{\leftarrow}(x) = Z(f - x)$ is an unbounded closed subset of $\mathbb{R}$ and so $Z(f - x)$ fails to be compact. Then $\mathrm{cl}_{\beta\mathbb{R}} Z(f - x) = Z(f^{\beta(\omega)} - x)$ where $Z(f^{\beta(\omega)} - x) \cap \beta\mathbb{R} \backslash \mathbb{R} \neq \varnothing$. Then the family

$$\{f^{\beta(\omega)}|^{\leftarrow}_{\beta\mathbb{R}\backslash\mathbb{R}}(x) : x \in \mathbb{R}^+\} \cup \{f^{\beta(\omega)^{\leftarrow}}(\infty)\}$$

forms an uncountable number of sets which partition $\beta\mathbb{R} \backslash \mathbb{R}$. The function $f^{\beta(\omega)}$ is real-valued on $[\beta\mathbb{R} \backslash \mathbb{R}] \backslash \{f^{\beta(\omega)^{\leftarrow}}(\infty)\}$.

Concepts review.

1. Suppose S is a topological space and T is a compact Hausdorff space. What does it mean to say that T is a compactification of S?

2. If S has a compactification, αS, what separation axiom is guaranteed to be satisfied by S?

3. Given a completely regular space S let $e : S \to \pi_{i\in I}[a_i, b_i]$ be the evaluation map on S induced by $C^*(S)$. Give a definition of the Stone-Čech compactification of S which involves this evaluation map.

4. What does it mean to say that the two compactifications of S, αS and γS, are equivalent compactifications?

5. If $\mathscr{C} = \{\alpha_i S : i \in I\}$ denotes the family of all compactifications of S. Define a partial ordering of $\mathscr{C}$.

6. Describe how we can use the function $\arctan : \mathbb{R} \to (-\pi/2, \pi/2)$ two construct a two-point compactification of $\mathbb{R}$.

7. If U is a subset of the topological space S, what does it mean to say that U is C^*-embedded in S?

8. If S is C^*-embedded in the compactification, αS, of S what can we say about αS?

9. Suppose S is completely regular and $g : S \to K$ is any continuous function mapping S into a compact Hausdorff space K. For which compactification, αS, does the following statement hold true: "the function g extends to a continuous function $g^* : \alpha S \to K$"?

10. Given two compactifications αS and γS of S, how is the function $\pi_{\gamma\to\alpha}\gamma S \to \alpha S$ defined?

11. Suppose S is locally compact and Hausdorff. Define the one-point compactification, ωS, of S.

12. State the *Urysohn extension theorem.*

13. What is the Stone-Čech compactification of the ordinal space $[0, \omega_1)$?

14. State a characterization of the pseudocompact property stated in this chapter.

15. Characterize the *locally compact* property of a space S in terms of its compactification αS of S.

16. Suppose T is a non-empty subset of a completely regular space S. What property must T satisfy in relation to S if $\mathrm{cl}_{\beta S} T = \beta T$?

17. We know the cardinalities $|\mathbb{N}| = |\mathbb{Q}| = \aleph_0$ and $|\mathbb{R}| = c$. What are the cardinalities of $\beta\mathbb{N}$, $\beta\mathbb{Q}$ and $\beta\mathbb{R}$?