19 / Paracompact topological spaces.

Abstract. *We will develop in this section some familiarity with the paracompact property. After giving a formal definition we provide a few examples and discuss its invariance properties. Finally we show that all metrizable spaces are paracompact.*

19.1 Paracompact topological spaces.

We now introduce a last important class of topological spaces closely related to the family of compact spaces. Its importance was discovered in the role it played in characterizations of metrizable spaces. We begin by introducing some new terminology. Note that some of what we present here may appear somewhat familiar since we have already (informally) introduced the concept of a "locally finite" family of sets before (on page 131). For convenience we formally reintroduce the concept here along with associated terms.

Definition 19.1 Let S be a topological space.

(a) Let $\mathscr{U} = \{U_i : i \in I\}$ be a family of subsets of S. We say that $\mathscr{U}$ is a *locally finite family of sets* if each point in S has a neighborhood which intersects only finitely elements of $\mathscr{U}$. [1]

(b) We say that $\mathscr{V} = \{V_j : j \in J\}$ *is a refinement of*, (or refines) the family, $\mathscr{U} = \{U_i : i \in I\}$, if every element of $\mathscr{V}$ is contained in some element of $\mathscr{U}$. If the elements of $\mathscr{V}$ are open, then we will more specifically say that $\mathscr{V}$ *is an open refinement* of $\mathscr{U}$. If $S \subseteq \cup\mathscr{V}$ then we say that $\mathscr{V}$ is *an open refinement cover of* S.

(c) Let $\mathscr{U}$ be a family of subsets of the topological space S. We say that $\mathscr{U}$ *has a locally finite open refinement* if there is a family, $\mathscr{V}$, of open subsets in S which both refines $\mathscr{U}$ and satisfies the locally finite property in S.

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[1] The word "neighborhood-finite" is sometimes used instead of "locally finite". A locally finite collection of subsets of a space S can be visualized as one whose elements do not cluster around any of its points.

Whether a family of sets is locally finite or not in the space, S, depends on the topology of S (since the definition refers to neighborhoods of S). Whether $\mathscr{V}$ refines $\mathscr{U}$ or not refers to a set-theoretic property which characterizes a particular relationship between between the elements of $\mathscr{V}$ and those of $\mathscr{U}$. *Open refinement* adds a particular topological characteristic to the elements of the refinement.

For example, suppose $S = [-3, 3]$ is equipped with the subspace topology. Consider the following family of subsets:

$$\begin{aligned}
\mathscr{S}_0 &= \{[-3,-1), (-2,0), (-1,1), (0,2), (1,3]\} \\
\mathscr{S}_1 &= \mathscr{S}_0 \cup \{(-1/2, 0)\} \\
\mathscr{S}_2 &= \mathscr{S}_0 \cup \{(-1/2, 1/2)\} \\
\mathscr{S}_3 &= \mathscr{S}_0 \cup \{(0, 1/n) : n \in \mathbb{N}\backslash\{0\}\} \\
\mathscr{S}_4 &= \{\left(\tfrac{n}{2}, \tfrac{n+1}{2}\right) : n = -6, -5, \ldots, 5\} \\
\mathscr{S}_5 &= \mathscr{S}_0 \cup \{(0, 1/2]\} \\
\mathscr{S}_6 &= \{[-3,-2), (-3,-3/2\}, (-2,0), (-1,1), (0,2), (1,3]\}
\end{aligned}$$

Verify that,

- The collection $\mathscr{S}_0$ is an open cover of S which is locally finite.
- The collection $\mathscr{S}_1$ is a locally finite open refinement of $\mathscr{S}_0$ which covers S.
- The collection $\mathscr{S}_2$ covers S, but is not an open refinement of $\mathscr{S}_0$ (since $(-1/2, 3/2)$ is not a subset of any element of $\mathscr{S}_0$.)
- The collection $\mathscr{S}_3$ covers S and is an open refinement of $\mathscr{S}_0$. But it is not locally finite since every open interval containing the point zero intersects infinitely many elements of $\mathscr{S}_3$.
- The collection $\mathscr{S}_4$ is an open refinement of $\mathscr{S}_0$ but it does not cover S (since $-3, 0, 3$, for example, are not contained in any of its elements.
- The collection $\mathscr{S}_5$ is a refinement of $\mathscr{S}_0$ which is locally finite but it is not an open refinement since $(0, 1/2]$ is not open.
- The collection $\mathscr{S}_6$ is an open refinement of $\mathscr{S}_0$ which covers S and is locally finite .

We now define the main subject of this section.

Definition 19.2 Let S be a topological space. The space S is said to be a *paracompact* space if, for any open cover $\mathscr{U}$ of S, there exists a locally finite open cover, $\mathscr{V}$, of S which refines $\mathscr{U}$.[2]

We alert the reader to the fact that, in some books, the Hausdorff property is incorporated into the formal definition of the paracompact property, in the sense that, for these authors, all paracompact spaces are hypothesized to be Hausdorff. In this book, a paracompact space is Hausdorff only when we explicitly state it as such.

Note that, if, for any open cover $\mathscr{U}$ of a space S, there is a finite subfamily, $\mathscr{F}$, of $\mathscr{U}$ which covers the space S, then by definition, $\mathscr{F}$ refines $\mathscr{U}$ (in the sense that "*... every element of the finite subfamily $\mathscr{F}$ is contained in some element of $\mathscr{U}$*") and, again by definition, $\mathscr{F}$ is locally finite in S (in the sense that "*... each point in S has a neighborhood which intersects only finitely elements (in fact, only one) of $\mathscr{F}$*").

We can then say that ...,

> *... if S is compact and $\mathscr{U}$ is an open cover, then $\mathscr{U}$ has a locally finite open refinement $\mathscr{F}$.*

Theorem 19.3 Any compact space is a paracompact space.

Proof: This statement is proven above.

Since all compact spaces are paracompact, we then know of a large family of topological spaces which are paracompact. We consider the following example of a non-compact paracompact space.

Example 1. Let S be an infinite space equipped with the discrete topology. Since $\mathscr{V} = \{\{x\} : x \in S\}$ is an open cover with no subcover, the space, S, is not compact. Show that S is, nevertheless, paracompact.

Solution: Let $\mathscr{U} = \{U_i : i \in I\}$ be an arbitrary open cover of S. To show that S is paracompact, it suffices to show that $\mathscr{U}$ has a locally

[2] Note that $\mathscr{V}$ satisfies four conditions: 1) $\mathscr{V}$'s elements must be open sets, 2) $\mathscr{V}$ covers S, 3) every element of $\mathscr{V}$ is a subset of an element of $\mathscr{U}$, 4) $\mathscr{V}$ must be locally finite.

finite open refinement. From the definition, the family $\mathscr{V} = \{\{x\} : x \in S\}$ is an open refinement of $\mathscr{U}$ (since every one of its elements, $\{x\}$, is a subset of some set, U_i, in $\mathscr{U}$). It then suffices to show that $\mathscr{V}$ is a locally finite family of sets. Let $x \in S$. Consider the open neighborhood, $B = \{x\}$, of x. The neighborhood B intersects only one element, $\{x\} \in \mathscr{V}$. So $\mathscr{V}$ is a locally finite family of sets in S.

So $\mathscr{V}$ is a locally finite open refinement of $\mathscr{U}$. We can conclude that S is paracompact.

In the following example we show that $\mathbb{R}$ is a paracompact space.

Example 2. Consider the space $S = \mathbb{R}$ equipped with the usual topology. We know that this metrizable space is unbounded and so is not compact. Show that S is paracompact.

Solution: Suppose we have an open cover, $\mathscr{U} = \{U_i : i \in I\}$, of S. We are required to construct an open refinement of $\mathscr{U}$ which is locally finite and covers S.

Firstly, we will construct an open refinement, $\mathscr{V}$, of $\mathscr{U}$ which covers S. For each $n \in \mathbb{N}\backslash\{0\}$, let $(-n, n)$ denote an open interval centered at 0 of radius n. Since, for each n, $[-n, n]$ is closed and bounded it is a compact subset of S. So, for each n, $[-n, n]$ has a finite open cover, say, $\mathscr{U}_n = \{U_i : i \in F_n\} \subseteq \mathscr{U}$. (Check!)

For example, if $n = 5$, the subset $[-5, 5]$ has a finite open cover, say, $\mathscr{U}_5 = \{U_i : i \in F_5\} \subseteq \mathscr{U}$. Then,

$$[-4, 4] \subset [-5, 5] \subseteq \cup\mathscr{U}_5$$

Then

$$[-5, 5]\backslash[-4, 4] \;\subseteq\; \cup\mathscr{U}_5\backslash[-4, 4] \;=\; \cup\{U_i\backslash[-4, 4] : i \in F_5\}$$

For each $i \in F_5$, let

$$V_i \;=\; U_i\backslash[-4, 4] \;\subseteq\; U_i$$

Then, for each $i \in F_5$, V_i is open in $\mathbb{R}$ and contained in U_i. Furthermore,

$$[-5, 5]\backslash[-4, 4] \;\subseteq\; \cup\{V_i : i \in F_5\}$$

If we let

$$\mathscr{V}_5 = \{V_i : i \in F_5\}$$

then $\mathscr{V}_5$ is a finite open cover of $[-5, 5]\backslash[-4, 4]$ which refines $\mathscr{U}_5$.

Generalizing from 5 to any natural number n, $\mathscr{V}_n$ is an open cover of $[-n,n]\backslash[-n+1,n-1]$ which refines $\mathscr{U}_n$.

If

$$\mathscr{V} = \cup\{\mathscr{V}_n : n \in \mathbb{N}\backslash\{0\}\}$$

then $\mathscr{V}$ is an open refinement of $\mathscr{U}$ which covers all of S.

We now claim that $\mathscr{V}$ is locally finite.

If $p \in S$, then $p \in [-n,n]\backslash[-n+1,n-1]$, for some natural number, n, and so p has a neighborhood which intersects at most finitely elements of $\mathscr{V}_n \subseteq \mathscr{V}$. So $\mathscr{V}$ is locally finite.

We have constructed the family, $\mathscr{V}$, which is both an open refinement of the open cover, $\mathscr{U}$, and is locally finite. So $S = \mathbb{R}$ is paracompact. We are done.

Example 3. Consider the space $S = \mathbb{R}^n$ equipped with the usual topology. We know that, for any n, this metrizable space is not compact. Show that S is paracompact.

Solution: To solve this we mimic the procedure in the previous example, replacing the interval $(-n,n)$ with the open ball, $B_n(0)$, centered at 0.

We know that Hausdorff compact spaces are normal. We have a similar result for Hausdorff paracompact spaces which states that Hausdorff paracompact spaces are guaranteed to be "normal" topological spaces.

In the proof of the following theorem we invoke the Lemma (6.17 on page 131) which we restate here, for convenience.

Lemma 6.17: *If $\mathscr{U} = \{V_i : i \in I\}$ is a locally finite collection of sets in S, then $\mathscr{U}^* = \{\text{cl}_S V_i : i \in I\}$ is also locally finite. Furthermore,*

$$\text{cl}_S[\cup\{V_i : i \in I\}] = \cup\{\text{cl}_S V_i : i \in I\}$$

Theorem 19.4 Let S be a Hausdorff paracompact topological space.

(a) The space, S, is a regular space.

(b) The space, S, is a normal space.

Proof: We are given that S is a Hausdorff paracompact topological space.

(a) We are required to show that S is regular.

Suppose H is a closed subset of S and $u \in S\backslash H$. It suffices to show that there is an open set, W, such that $H \subseteq W \subseteq \text{cl}_S W \subseteq S\backslash\{u\}$.

Since S is Hausdorff and H is closed, then, for each $x \in H$, there exists an open neighborhood, U_x, of x such that $u \notin \text{cl}_S U_x$. Now,

$$\mathscr{U} = \{U_x : x \in H\} \cup \{S\backslash H\}$$

forms an open cover of S. Since S is paracompact and $\mathscr{U}$ is an open cover of S, there exists a locally finite collection of open sets

$$\mathscr{V} = \{V_i : i \in I\} \cup \{V\}$$

which covers S and refines the open cover, $\mathscr{U}$ (where each $V_i \subseteq U_{y_i}$, for some $y_i \in H$, and $V \subseteq S\backslash H$).

Then, for each i,

$$V_i \subseteq \text{cl}_S V_i \subseteq \text{cl}_S U_{y_i} \subseteq S\backslash\{u\} \quad \text{(For some } y_i \in H\text{)}$$

Let $W = \cup\{V_i : i \in I\}$. Then, by the Lemma 6.17, (which states that the union of a locally finite collection of closed sets is closed)

$$\text{cl}_S W = \cup\{\text{cl}_S V_i : i \in I\}$$

It follows that

$$H \subseteq W = \cup\{V_i : i \in I\} \subseteq \text{cl}_S W = \cup\{\text{cl}_S V_i : i \in I\} \subseteq S\backslash\{u\}$$

So S is regular. As required.

(b) We are now required to prove that S is a normal space. Let H be a closed set in S. To prove the normal property holds we replace the point u in the proof of part (a) with a closed set, F, and invoke the regular property and mimic the steps in the proof above.

Suppose H is a closed subset of S and F is a closed subset such that $F \in S\backslash H$.

For each $x \in H$, there exists an open neighborhood, U_x such that $F \cap \text{cl}_S U_x = \varnothing$ (since we have shown that S satisfies the regular property).

Now,

$$\mathscr{U} = \{U_x : x \in H\} \cup \{S\backslash H\}$$

forms an open cover of S (with $F \subseteq S\backslash H$). By hypothesis, there exists a locally finite collection of open sets

$$\mathscr{V} = \{V_i : i \in I\} \cup \{V\}$$

which refines $\mathscr{U}$, where each V_i is a subset of some U_x and $V \subseteq S\backslash H$.
Then, for each i,

$$V_i \subseteq \text{cl}_S V_i \subseteq \text{cl}_S U_x \subseteq S\backslash F \quad \text{(For some } x \in H.)$$

Again, by Lemma 6.17, $\cup\{\text{cl}_S V_i : i \in I\} = \text{cl}_S(\cup\{V_i : i \in I\})$.
Since $\text{cl}_S(\cup\{V_i : i \in I\}) = \cup\{\text{cl}_S V_i : i \in I\} \subseteq (\cup\{\text{cl}_S U_x : x \in H\}) \subseteq S\backslash F$, S is normal, the desired property.

The following characterization (one of many) will prove useful later in the text.

Theorem 19.5 A regular space S is paracompact if and only if every open cover has a closed locally finite refinement which covers S.

Proof : We are given that S is a regular space.

($\Rightarrow$) This direction is straightforward. Suppose S is paracompact. Let $\mathscr{U}$ be a an open cover of S.

If $p \in U_p \in \mathscr{U}$ then, since S is regular, there exists an open subset H_p such that $p \in H_p \subseteq \text{cl}_S H_p \subseteq U_p$. Then the collection $\mathscr{H} = \{H_p : p \in S\}$ is an open refinement of $\mathscr{U}$ such that, the closure of its elements refines $\mathscr{U}$. Since $\mathscr{H}$ covers S and S is paracompact, there is an open locally finite refinement $\mathscr{W} = \{W_p : p \in S\}$ of $\mathscr{H}$ such that $W_p \subseteq \text{cl}_S W_p \subseteq \text{cl}_S H_p \subseteq U_p \in \mathscr{U}$. Then $\mathscr{W}^* = \{\text{cl}_S W_p : p \in S\}$ refines $\mathscr{U}$.

We claim that $\mathscr{W}^*$ is locally finite. Let $x \in S$. Since $\mathscr{W}$ is locally finite there exists an open neighborhood D of x such that D meets at most finitely elements of $\mathscr{W}$. If $W \cap D = \varnothing$ then, since D is open, $\text{cl}_S W \cap D = \varnothing$. So $\mathscr{W}^*$ is also locally finite. We conclude that $\mathscr{U}$ has a closed locally finite refinement which covers S. As required.

($\Leftarrow$) Suppose $\mathscr{U}$ is an open cover of S. Then, by hypothesis, there is a closed locally finite set $\mathscr{H}$ which refines $\mathscr{U}$ and covers S.

For each $x \in S$, there exists an open neighborhood V_x which meets finitely many elements of $\mathscr{H}$. Let $\mathscr{V} = \{V_x : x \in S\}$. Then $\mathscr{V}$ covers S. By hypothesis, there is a closed locally finite set, $\mathscr{C}$, which refines

$\mathscr{V}$ and covers S. Note that, for each $C \in \mathscr{C}$, $C \subseteq V_x$ and since each V_x meets only finitely many elements of $\mathscr{H}$

$$\ldots \text{each } C \text{ meets finitely many elements of } \mathscr{H}$$

Since $\mathscr{H}$ refines the open cover, $\mathscr{U}$, for each $H \in \mathscr{H}$, there is an open set U_H in $\mathscr{U}$ such that

$$H \subseteq U_H \in \mathscr{U}$$

Each $C \in \mathscr{C}$ meets only finitely many H's,

$+$ each H can meet only finitely many U's in $\mathscr{U}$,

$\Rightarrow$ each C can only meet finitely many U's in $\mathscr{U}$.

For each $H \in \mathscr{H}$, let

$$E_H = S \backslash \cup \{C \in \mathscr{C} : C \subseteq S \backslash H\}$$

See that, since $\mathscr{C}$ is locally finite, $\cup\{C \in \mathscr{C} : C \subseteq S \backslash H\}$ is closed hence, E_H is open.

Claim #1. We claim that, for each $H \in \mathscr{H}$, $H \subseteq E_H$. Suppose $y \in H$.

$$\begin{aligned} y \notin E_H &\Rightarrow y \in \cup\{C \in \mathscr{C} : C \subseteq S \backslash H\} \\ &\Rightarrow y \in C, \text{ for some } C \in \mathscr{C} \text{ such that } C \subseteq S \backslash H \\ &\Rightarrow y \notin H. \quad \text{(Contradiction!)} \end{aligned}$$

So, for each $H \in \mathscr{H}$, $H \subseteq E_H$, as claimed.

If $x \in S$, then $x \in H$ for some H, $x \in U_H$, so

$$x \in E_H \cap U_H$$

So $\mathscr{D}$ covers S. Since, for each $H \in \mathscr{H}$, $E_H \cap U_H \subseteq U_H \in \mathscr{U}$, then $\mathscr{D}$ refines $\mathscr{U}$.

Then the collection

$$\mathscr{D} = \{E_H \cap U_H : H \in \mathscr{H}\}$$

forms an open refinement of $\mathscr{U}$ which covers S.

If we can show that $\mathscr{D}$ is locally finite, then S is paracompact and we are done.

Claim #2. The set $\mathscr{D}$ is locally finite.

Let $p \in S$. Since $\mathscr{C}$ is locally finite, p has a neighborhood, B, which intersects finitely many elements, $\{C_i : i = 1, \ldots, k\} \subseteq \mathscr{C}$. We will show that B meets finitely many elements of $\mathscr{D}$.

Since $\mathscr{C}$ is a cover of S, then

$$B \subseteq \cup\{C_i : i = 1, \ldots, k\}$$

It suffices to show that each element of $\{C_i : i = 1, \ldots, k\}$ intersects $E_H \cap U_H$ for only finitely many H.[3]

See that if $C_j \cap [E_{H_1} \cap U_{H_1}] \neq \varnothing$ then $C_j \cap U_{H_1} \neq \varnothing$.

Recall that C in $\mathscr{C}$ can intersect at most finitely many U's in $\mathscr{U}$.

We conclude that B intersects at most finitely many elements of $\mathscr{D}$.

So $\mathscr{D}$ is locally finite, as claimed.

We conclude that $\mathscr{D}$ is an open refinement of $\mathscr{U}$ which both covers S and is locally finite in S. So the metrizable space, S, is paracompact.

Theorem 19.6 Let S be a paracompact topological space and F be a closed subset of S. Then F inherits the paracompact property from S.

Proof: We are given that S is a paracompact topological space and F is closed in S. We are required to show that F is also paracompact.

Let $\mathscr{U} = \{U_\alpha : \alpha \in I\}$ be an open cover of F. Then, for each α, $U_\alpha = V_\alpha \cap F$, for some open subset V_α of S. We then obtain an open cover

$$\mathscr{V} = \{S \backslash F\} \cup \{V_\alpha : \alpha \in I\}$$

of the paracompact space, S. Then there is a locally finite open refinement of $\mathscr{V}$, say, $\mathscr{W} = \{W_\alpha : \alpha \in I\}$, where $W_\alpha \subseteq V_\alpha$, which covers S. Then $\mathscr{D} = \{W_\alpha \cap F : \alpha \in I\}$ covers F, where $W_\alpha \subseteq U_\alpha$, and so is a locally finite open refinement of $\mathscr{U}$. Then F is paracompact.

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[3] For, if each element of $\{C_i : i = 1, \ldots, k\}$ meets finitely many elements of $\mathscr{D}$, B can only meet finitely elements of $\mathscr{D}$)

The next example involves the product of two spaces, one of which is paracompact.

Example 4. Show that the product of a Hausdorff paracompact space with a Hausdorff compact space is paracompact.

Solution: We are given that S is a paracompact Hausdorff space and T is a Hausdorff compact space. Let $\mathscr{U} = \{U_\alpha : \alpha \in I\}$ be an open cover of the product space $S \times T$. For $x \in S$, the set

$$\{x\} \times T \;\subseteq\; \cup\{\{x\} \times \pi_2[U_\alpha] : \alpha \in I\}$$

has a finite subcover, say $\{\{x\} \times \pi_2[U^x_{\alpha_i}] : i = 1, \ldots, n_x\}$.

For each x in S, we can choose an open neighborhood, V_x, of x such that

$$\{V_x \times \pi_2[U^x_{\alpha_i}] : i = 1, \ldots, n_x\} \subseteq \cup\{U^x_{\alpha_i} : i = 1, \ldots, n_x\}$$

See that the set $\mathscr{V} = \{V_x : x \in S\}$ forms an open cover of paracompact S, so $\mathscr{V}$ has a locally finite open refinement, $\mathscr{W}$. For any $W \in \mathscr{W}$, there is an $x \in S$ such that $W \subseteq V_x \in \mathscr{V}$ and so $W \times \pi_2[U^x_{\alpha_i}] \subseteq U_\alpha$ for some α.

Let $(x, y) \in S \times T$, there exists an open neighborhood D of x such that D intersects only finitely many W's in $\mathscr{W}$. So

$$\mathscr{U}^* = \{W \times \pi_2[U^x_{\alpha_i}] : i = 1, \ldots, n_x\} : W \in \mathscr{W}\}$$

is a locally finite open refinement of $\mathscr{U}$. So $S \times T$ is paracompact.

19.2 A non-paracompact space.

We now present an example of a space which is not paracompact.

Example 5. A standard, non-trivial, example of a Hausdorff non-paracompact space is the deleted Tychonoff plank, $S = [0, \omega_1] \times [0, \omega_0] \setminus \{(\omega_1, \omega_0)\}$. On page 220, we showed that the space S is not normal. Since Hausdorff paracompact spaces have been shown to be normal, then S cannot be paracompact.

Also, the Moore plane was shown to be completely regular but not normal. So this is another example of a non-paracompact space.

19.3 Topic: Metrizability and paracompactness.

Any result which can bring us a step closer to a characterization of metrizable spaces is considered to be important and worth the effort required to study the proofs of related statements. The following definitions and technical results, all involving the paracompact property, will lead to the threshold of a proof of the statement, "Metrizable spaces are paracompact spaces".

Studying these will help the reader develop a deeper understanding of the paracompactness property. This is the main reason why we present it now. We begin by providing the following definition.

Definition 19.7 Let S be a topological space and $\mathscr{U}$ be a subfamily of $\mathscr{P}(S)$. If

$$\mathscr{U} = \cup\{\mathscr{U}_n : n \in \mathbb{N}\}$$

where each $\mathscr{U}_n$ is a locally finite subset of $\mathscr{P}(S)$, then we say that $\mathscr{U}$ is σ-locally finite.

We will begin by listing the four statements which will lead to fifth statement "If a space is metrizable, then it is paracompact".

1) We have already proven the first statement in Theorem 19.5:

A regular space S is paracompact if and only if every open cover of S has a closed locally finite refinement which covers S.

2) The next statement is Lemma 19.8 involves the notion of the σ-locally finite property:

If S is a metrizable space and $\mathscr{U}$ is an open cover of S then there is a σ-locally finite open cover, $\mathscr{E}$, which refines $\mathscr{U}$.

3) The next statement is Lemma 19.9:

If S is a metrizable space and $\mathscr{U}$ is an open cover of S, then there is a locally finite refinement, $\mathscr{C}$, which covers S. (The elements of $\mathscr{C}$ are not necessarily open.)

4) The next statement is Lemma 19.10:

If S is a metrizable space and $\mathscr{U}$ is an open cover of S, then there is a closed locally finite refinement, $\mathscr{C}$, which covers S.

Finally, we attain our objective with Theorem 19.11:

If S is a metrizable space then S is paracompact.

Lemma 19.8 Suppose S is a metrizable space and $\mathscr{U}$ is an open cover of S. Then there is an open cover, $\mathscr{E}$, which both refines $\mathscr{U}$ and is σ-locally finite.[4]

Proof : Let S be a metrizable topological space. Then there is a metric, ρ, which allows us to express S as a metric space, (S, ρ). Suppose $\mathscr{U}$ is an open cover of S.

We will construct a subfamily, $\mathscr{E}$, of $\mathscr{P}(S)$ such that

1) $\mathscr{E}$ is an open cover of S,
2) $\mathscr{E}$ refines $\mathscr{U}$,
3) $\mathscr{E} = \cup\{\mathscr{E}_n : n \in \mathbb{N}\backslash\{0\}\}$ where each $\mathscr{E}_n$ is a locally finite collection of open sets.

We will index the elements of the open cover, $\mathscr{U}$, with ordinals Ω: $\mathscr{U} = \{U_\alpha : \alpha \in \Omega\}$. That is,

$$\mathscr{U} = \{U_0, U_1, \ldots, U_{\omega_0}, U_{\omega_0+1} \ldots, U_{\omega_1}, U_{\omega_1+1}, U_{\omega_1+2}, \ldots, \}^5$$

Step 1: Construction of the collection of sets, $\{\mathscr{E}_n : n \in \mathbb{N}\backslash\{0\}\}$.

Throughout Step 1, the natural number n in $\mathbb{N}\backslash\{0\}$ is fixed, as U_α ranges through $\mathscr{U}$.

For each $U_\gamma \in \mathscr{U}$, let

$$S_n(U_\gamma) = \{x \in S : B_{1/n}(x) \subseteq U_\gamma\}$$

Let

$$T_n(U_\gamma) = S_n(U_\gamma)\backslash\cup\{U_\alpha : \alpha < \gamma\}$$

Now see that $T_n(U_\gamma) \subseteq U_\gamma$.[6] If we repeat this for each element, U_α, of $\mathscr{U}$, then

$$\mathscr{T}_n = \{T_n(U_\alpha) : \alpha \in \Omega\}$$

is a refinement of $\mathscr{U}$.

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[4]Note that $\mathscr{E}$ is "σ-locally finite" (not locally finite) so we cannot conclude immediately that metrizable spaces are paracompact. This will come later.

[5]Here we are invoking the *Well-ordering theorem* which permits such an ordering. It is a statement which is equivalent to the *Axiom of choice*.

[6]For, if $x \in T_n(U_\gamma)$, then $x \in B_{1/n}(x) \subseteq U_\gamma$.

Claim #1. We claim that the elements of $\mathscr{T}_n$ are pairwise disjoint.

Proof of claim: Consider $T_n(U_\beta)$ and $T_n(U_\gamma)$ where $\beta < \gamma$. Choose element $c \in T_n(U_\gamma)$. Then, for any point $b \in T_n(U_\beta)$, $b \in S_n(U_\beta)$. So $b \in B_{1/n}(b) \subseteq U_\beta$. Since $c \in T_n(U_\gamma)$ and $\beta < \gamma$, $c \notin U_\beta$. So $c \notin B_{1/n}(b)$. So $c \notin T_n(U_\beta)$. We have shown that,

$$b \in T_n(U_\beta) \text{ and } c \in T_n(U_\gamma) \Rightarrow c \notin B_{1/n}(b) \quad (*)$$

So the elements of $\mathscr{T}_n$ are pairwise disjoint, *as claimed*.

Even if $\mathscr{T}_n$ is a refinement of $\mathscr{U}$, its elements may not be open, so we have more work to do.

We will construct an open set $E_n(U_\gamma)$ such that

$$T_n(U_\gamma) \subseteq E_n(U_\gamma) \subseteq U_\gamma$$

For, $U_\gamma \in \mathscr{U}$, let

$$E_n(U_\gamma) = \cup\{B_{1/3n}(x) : x \in T_n(U_\gamma)\}$$

We repeat this for each element, U_α, of $\mathscr{U}$. We define the set

$$\mathscr{E}_n = \{E_n(U_\alpha) : \alpha \in \Omega\}$$

where each element, $E_n(U_\alpha)$, of $\mathscr{E}_n$ is an open subset of S. This completes Step 1.

Each element of the collection

$$\{\mathscr{E}_1, \mathscr{E}_2, \mathscr{E}_3, \ldots, \}$$

is thus defined.

Step 2. We will now show that each element of, $\{\mathscr{E}_1, \mathscr{E}_2, \mathscr{E}_3, \ldots, \}$ is a locally finite collection of open subsets of S.

By definition, for each n, $\mathscr{E}_n$ is a collection of open sets.

Claim #2. We *claim* that the elements of $\mathscr{E}_n = \{E_n(U_\alpha) : \alpha \in \Omega\}$ are pairwise disjoint. That is, given n, if $\beta < \gamma$, $E_n(U_\beta) \cap E_n(U_\gamma) = \varnothing$.

Proof of claim: Consider the pair $E_n(U_\beta)$ and $E_n(U_\gamma)$ where $\beta < \gamma$. Let $v \in E_n(U_\gamma)$ and u be any element in $E_n(U_\beta)$. We will show that $v \neq u$.

$$\begin{aligned} u \in E_n(U_\beta) = \cup\{B_{1/3n}(x) : x \in T_n(U_\beta)\} \quad &\Rightarrow \quad u \in B_{1/3n}(b) \text{ for some } b \in T_n(U_\beta) \\ v \in E_n(U_\gamma) = \cup\{B_{1/3n}(x) : x \in T_n(U_\gamma)\} \quad &\Rightarrow \quad v \in B_{1/3n}(c) \text{ for some } c \in T_n(U_\gamma) \\ b \in T_n(U_\beta) \text{ and } c \in T_n(U_\gamma) \quad &\Rightarrow \quad c \notin B_{1/n}(b) \quad \text{(By (*).)} \end{aligned}$$

Then

$$\begin{array}{rcl} 1/n & \leq & \rho(b,c) \\ & \leq & \rho(b,v)+\rho(c,v) \\ & < & [\rho(b,u)+\rho(u,v)]+1/3n \\ & < & 1/3n+\rho(u,v)+1/3n \\ & = & \rho(u,v)+2/3n \end{array}$$

$$\text{implies} \quad \rho(u,v) \ > \ 1/3n \qquad (**)$$

So $v \notin B_{1/3n}(u)$.

We have shown that,

$$u \in E_n(U_\beta) \text{ and } v \in E_n(U_\gamma) \Rightarrow v \notin B_{1/3n}(u) \quad (***)$$

So $v \neq u$ and $u \in E_n(U_\beta)$ then $v \notin E_n(U_\beta)$. We conclude, if $\beta < \gamma$, $E_n(U_\beta) \cap E_n(U_\gamma) = \varnothing$; this proves *Claim* #2.

Claim #3. We claim that each $\mathscr{E}_n$ is locally finite.

Proof of claim: Let $z \in S$. By Claim #2, z belongs to at most one element of $\mathscr{E}_n$, say $E_n(U_\gamma) = \cup\{B_{1/3n}(x) : x \in T_n(U_\gamma)\}$. Say $z \in B_{1/3n}(y)$. By (**) the distance between points in different sets in $\mathscr{E}_n$ is larger than $1/3n$. So the ball $B_{1/6n}(z)$ intersects only $E_n(U_\gamma)$. Then $\mathscr{E}_n$ is locally finite, as claimed.

This completes Step 2.

Step 3. We define

$$\mathscr{E} = \cup\{\mathscr{E}_n : n \in \mathbb{N}\backslash\{0\}\} \ = \ \mathscr{E}_1 \cup \mathscr{E}_2 \cup \mathscr{E}_3 \cup \cdots \cup \mathscr{E}_n \cup \cdots$$

We now verify that the collection $\mathscr{E}$ of subsets satisfies all the required properties: 1) Each element of $\mathscr{E}$ is open in S. (Already verified above.) 2) $\mathscr{E}$ is σ-locally finite. 3) $\mathscr{E}$ covers S. 4) $\mathscr{E}$ refines $\mathscr{U}$.

1) The elements of $\mathscr{E}$ are of the form $E_n(U_\alpha)$. We have already verified that these are open subsets of S.

2) We have verified that each $\mathscr{E}_n$ is locally finite. So $\mathscr{E}$ is σ-locally finite.

3) The collection $\mathscr{E}$ is an open cover of S: Let $y \in S$. We must show that y must belong to some member, $E_m(U_\mu) \in \mathscr{E}_m$, of $\mathscr{E}$.

Let μ be the least ordinal such that $y \in U_\mu \in \mathscr{U}$.[7] Since U_μ is open we can choose a natural number m such that $B_{1/m}(y) \subseteq U_\mu$. Then

[7] The ordinal μ exists since the ordinals are "well-ordered".

$y \in S_m(U_\mu) = \{x \in S : B_{1/m}(x) \subseteq U_\mu\}$. Since μ is the least ordinal such that $y \in U_\mu$, then $y \in T_m(U_\mu) = S_m(U_\mu)\backslash \cup\{U_\alpha : \alpha < \mu\}$. By definition, $B_{1/3m}(y) \subseteq E_m(U_\mu) = \cup\{B_{1/3m}(x) : x \in T_n(U_\mu)\}$, so $y \in E_m(U_\mu) \in \mathscr{E}_m \subseteq \mathscr{E}$. Then each element of S belongs to some element of $\mathscr{E}$. So $\mathscr{E}$ is an open cover of S.

4) We *claim* that $\mathscr{E}$ refines $\mathscr{U}$.

$$\begin{aligned}
u \in E_n(U_\gamma) &\Rightarrow u \in B_{1/3n}(c) \text{ for some } c \in T_n(U_\gamma)\\
T_n(U_\gamma) = S_n(U_\gamma)\backslash \cup\{U_\alpha : \alpha < \gamma\} &\Rightarrow c \in S_n(U_\gamma)\\
S_n(U_\gamma) = \{x \in S : B_{1/n}(x) \subseteq U_\gamma\} &\Rightarrow B_{1/n}(c) \subseteq U_\gamma\\
u \in B_{1/3n}(c) \subseteq B_{1/n}(c) &\Rightarrow u \in U_\gamma
\end{aligned}$$

Then $E_n(U_\gamma) \subseteq U_\gamma$. Hence $\mathscr{E}$ is an open refinement of $\mathscr{U}$, *as claimed.*

We are done.

The next theorem takes us a step closer to the statement "Metrizable spaces are paracompact spaces". The reader will recognize the use of techniques similar to the ones used in the proof of the Lemma 19.8.

Lemma 19.9 Suppose the space S is metrizable and $\mathscr{U}$ is an open cover of S. Then $\mathscr{U}$ has a refinement, $\mathscr{C}$, which both covers S and is locally finite.[8]

Proof: Suppose S is a metrizable space and $\mathscr{U}$ is an open cover of S. By Lemma 19.8 there is an open cover, $\mathscr{E}$, which both refines $\mathscr{U}$ and is σ*-locally finite*. Let

$$\mathscr{E} = \cup\{\mathscr{E}_n : n \in \mathbb{N}\backslash\{0\}\}$$

be such an open cover where each $\mathscr{E}_n = \{E_{n(i)} : i \in I_n\}$ is a locally finite collection of open sets which refines $\mathscr{U}$. For each $m \in \mathbb{N}\backslash\{0\}$, let

$$V_m = \cup\{E_{m(i)} : i \in I_m\} = \cup\, \mathscr{E}_m$$

For $n \in \mathbb{N}\backslash\{0\}$, and $i \in I_n$ let

$$S_n(E_{n(i)}) = E_{n(i)} \backslash \cup\{V_m : m < n\} = E_{n(i)} \backslash \cup \left\{\cup\{E_{m(i)} : i \in I_m\} : m < n\right\}$$

[8] Note that the members of $\mathscr{C}$ need not be open in S.

We are required to construct a collection of sets, $\mathscr{C}$, which covers S, refines $\mathscr{E}$ and is locally finite. (Note that the members of $\mathscr{C}$ need not be open.)

Let

$$\mathscr{C}_n = \{S_n(E_{n(i)}) : i \in I_n\}$$

Since $S_n(E_{n(i)}) \subseteq E_{n(i)} \in \mathscr{E}_n$, then $\mathscr{C}_n$ refines $\mathscr{E}_n$.

Let

$$\mathscr{C} = \cup\{\mathscr{C}_n : n \in \mathbb{N}\backslash\{0\}\}$$

Since each $\mathscr{C}_n$ refines $\mathscr{E}_n$ and each $\mathscr{E}_n$ refines $\mathscr{U}$, then $\mathscr{C}$ is a refinement of $\mathscr{U}$

Claim A: It is claimed that $\mathscr{C}$ covers S.

Proof of claim: Let $p \in S$. We know that $\mathscr{E} = \cup\{\mathscr{E}_n : n \in \mathbb{N}\backslash\{0\}\}$ is an open cover of S. Let $k = \min\{n : p$ belongs to some element of $\mathscr{E}_n\}$. Let j be such that $p \in E_{k(j)} \in \mathscr{E}_k$. Then $p \in S_k(E_{k(j)}) \in \mathscr{C}_k \subseteq \mathscr{C}$. Then p belongs to some element of $\mathscr{C}$. So $\mathscr{C}$ covers S. This establishes claim A.

Claim B: It is claimed that for $p \in S$, p belongs to a neighborhood $B \subseteq E_{k(j)}$ which meets at most finitely many elements of $\mathscr{C}$. That is, $\mathscr{C}$ is a locally finite collection of sets in S.

Proof of claim: Let $p \in S$.

Let $k = \min\{n : p$ belongs to some element of $\mathscr{E}_n\}$ (as defined in the proof of claim 1). Then $p \in S_k(E_{k(i)}) \subseteq E_{k(i)}$, for some $i \in I_k$.

Recall that, for each n, $\mathscr{E}_n$ is locally finite. So, for each n, there exists a neighborhood, B_n, of p which intersects only finitely many elements in $\mathscr{E}_n = \{E_{n(i)} : i \in I_n\}$.

Because of this, this B_n can only intersect finitely may elements of $\mathscr{C}_n = \{S(E_{n(i)}) : i \in I_n\}$ for, whenever it intersects $S(E_{n(i)}) \subseteq E_{n(i)}$, it intersects $E_{n(i)}$. Also, if $n > k$, $E_{k(j)} \cap S_n(E_{n(i)}) = \varnothing$ (by definition of $S_n(E_{n(i)})$).

Let $B = \cap\{B_n : n \in \{1, 2, \ldots, k\}\}$. Then $p \in B$ and so $B \cap E_{k(i)}$ is an open neighborhood of p which meets at most finitely many elements of $\mathscr{C}$ (these are in $\cup\{\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_k,\}$). So $\mathscr{C}$ is locally finite.

Then the collection, $\mathscr{C}$, refines $\mathscr{U}$, is locally finite and covers S. As required.

Lemma 19.10 Suppose the space S is metrizable and $\mathscr{U}$ is an open cover of S. Then $\mathscr{U}$ has a refinement of *closed* sets which both covers S and is locally finite.

Proof : Suppose S is a metrizable space and $\mathscr{U}$ is an open cover of S. Since metrizable spaces are regular (by Theorem 9.19), then S is regular.

Then, for each $U \in \mathscr{U}$ and for each $x \in S$, there is some open neighborhood, B_x, of x such that $\mathrm{cl}_S B_x \subseteq U$. Then $\mathscr{B} = \{B_x : x \in S\}$ is an open cover of S which refines $\mathscr{U}$ where $B_x \subseteq U$ implies $\mathrm{cl}_S B_x \subseteq U$.

By Theorem 19.9, there is a collection of sets, $\mathscr{C} = \{C : C \in \mathscr{C}\}$, which refines $\mathscr{B}$, covers S and is locally finite. By Lemma 6.17, if $\mathscr{C}$ locally finite, then $\mathscr{C}^* = \{\mathrm{cl}_S C : C \in \mathscr{C}\}$ is also locally finite. Then the family $\mathscr{C}^*$ covers S (since $\mathscr{C}$ covers S) and is a refinement of $\mathscr{U}$ (since $\mathscr{C}$ refines $\mathscr{U}$).

So $\mathscr{C}^*$ is the desired refinement of $\mathscr{U}$ whose members are closed and which covers S.

Example 6. Suppose $\mathscr{F}$ is a locally finite collection of closed subsets in $\mathscr{P}(S)$. Show that $\cup\{F : F \in \mathscr{F}\}$ is a closed subset of S.

Solution: We are given that each $F \in \mathscr{F}$ is closed where $\mathscr{F}$ is locally finite. Let $M = \cup\{F : F \in \mathscr{F}\}$. From Lemma 6.17, we know that, if $\mathscr{F}$ is locally finite, $\{\mathrm{cl}_S F : F \in \mathscr{F}\}$ is locally finite and $\mathrm{cl}_S M = \cup\{\mathrm{cl}_S F : F \in \mathscr{F}\}$. Since

$$\cup\{\mathrm{cl}_S F : F \in \mathscr{F}\} = \cup\{F : F \in \mathscr{F}\}$$

then $M = \mathrm{cl}_S M$. So M is closed in S.

We are finally set for the main result of this section.

Theorem 19.11 If S is a metrizable space, then S is a Hausdorff paracompact space.

Proof : Suppose S is a metrizable space. Then S is Hausdorff. Suppose $\mathscr{U}$ is an open cover of S. By Theorem 19.9, there exists a family of sets, $\mathscr{H} = \{H : H \in \mathscr{H}\}$, which refines $\mathscr{U}$, covers S and is locally finite.

Since $\mathscr{H}$ is locally finite, for each $x \in S$ we can find open a neighborhood, V_x, of x which intersects at most finitely many elements of $\mathscr{H}$. Let

$$\mathscr{V} = \{V_x : x \in S\}$$

be the collection of all such sets.

Since $\mathscr{V}$ is an open cover of S, Lemma 19.10 applies: There is a collection of *closed* sets,

$$\mathscr{C} = \{C : C \in \mathscr{C}\}$$

which refines $\mathscr{V}$, covers S and is locally finite.

By Theorem 19.5, since metrizable spaces are regular, S is paracompact.

A complete characterization of metrizability will be presented in the chapter titled *Metrizability* further on in the text where we will prove that "A space S is metrizable if and only if S is a regular space which has a σ-locally finite base of open sets".

Concepts review.

1. What does it mean to say that $\mathscr{U}$ is a *locally finite* family of subsets of a space S?
2. What does it mean to say that the family of subsets, $\mathscr{V}$, is an *open refinement* of the family $\mathscr{U}$?
3. What does it mean to say that the family of subsets, $\mathscr{V}$, is a *locally finite open refinement* of the family $\mathscr{U}$?
4. Define the paracompact property of a topological space.
5. What class of topological spaces has elements which are guaranteed to be paracompact?
6. Provide an example of a paracompact space and briefly summarize arguments which confirms your answer.
7. Provide an example of a non-paracompact space.
8. If a paracompact space is Hausdorff what other separation axioms does it satisfy?

9. If $f : S \to T$ is a function mapping the paracompact space S into T, what properties must be satisfied by f if we want to guarantee that $f[S]$ is paracompact?

10. If S is a paracompact space what kind of subsets of S will share this property?