Appendix B.

Example 19 of Chapter 5. For each $k \in \mathbb{Z}$ and $n \in \mathbb{N}$, let

$$B(n,k) = \{y \in \mathbb{Z} : y = mn + k, m \in \mathbb{Z}\}$$

Let $\mathscr{B} = \{B(n,k) : k \in \mathbb{Z}, n \in \mathbb{N}\}$. Show that $\mathscr{B}$ forms an open base for some topology on $\mathbb{Z}$.

Solution: To show that the collection $\mathscr{B}$ of subsets of $\mathbb{Z}$, forms a base for some topology τ on $\mathbb{Z}$ we will show that it satisfies the two conditions stated in the Theorem 5.4.

Step 1. We first verify that $\mathbb{Z} = \cup\mathscr{B}$: Suppose $z \in \mathbb{Z}$ and $n \in \mathbb{N}$. Then $z = (0)n + z \in B(n,z)$ so $z \in \cup\{B(n,k) : k \in \mathbb{Z}, n \in \mathbb{N}\}$. Then

$$\mathbb{Z} = \cup\mathscr{B} = \cup\{B(n,k) : k \in \mathbb{Z}, n \in \mathbb{N}\}$$

Step 2. To show that $\mathscr{B}$ is a base for some topology on $\mathbb{Z}$, it now suffices to show that, if $x \in A \cap D$ (where $A, D \in \mathscr{B}$), there exists $C \in \mathscr{B}$ such that $x \in C \subseteq A \cap D$.

Let $r \in B(n,a) \cap B(q,d)$.

Then $r = tn + a = hq + d$ for some integers t and h. Note that for any integer r and natural number n, there are many ways of expressing r in the form $r = kn + a$. For example, if $k = k_1 + k_2 + k_3$ and $r = kn + a \in B(n,a)$ then

$$r = k_1 n + [(k_2 + k_3)n + a] \ \in \ B(n, [(k_2 + k_3)n + a])$$

But if, for the given integer r, we want the remainder a to be restricted to the set $\{0, 1, 2, 3, \ldots, n-1\}$ there can be only one integer value for k that will satisfy the equation $r = kn + a$. For an integer r and natural number n, we will denote the remainder whose value is in $\{0, 1, 2, \ldots, n-1\}$ by a_r.[1]
Given $r = tn + a \in B(n,a)$ where $t = p + s$, we can always write

$$r = pn + (sn + a_r) \ \in \ B(n, sn + a_r)$$

If $a = a_r$ we just let $s = 0$ and $p = t$.

[1]For example if $n = 5$ and $r = -63$,

$$\begin{aligned} r = -63 &= (-13)(5) + 2 \ \in \ B(5,2) \\ r = -63 &= (-12)(5) + -3 \ \in \ B(5,-3) \\ r = -63 &= (-14)(5) + 7 \ \in \ B(5,7) \end{aligned}$$

For a given integer $r \in B(n,a)$ we compare $B(n,a)$ and $B(n,a_r)$:

$$\begin{array}{lcl} x \in B(n,a) & \Rightarrow & x = tn + a = kn + sn + a_r = (k+s)n + a_r \ \in \ B(n,a_r) \\ x \in B(n,a_r) & \Rightarrow & x = tn + a_r = kn + (a - sn) = kn + a \ \in \ B(n,a) \end{array}$$

So, if $r \in B(n,a)$,

$$B(n,a) = B(n,a_r) \qquad (*)$$

In a similar fashion, if $r \in B(q,d)$, we obtain

$$B(q,d) = B(q,d_r) \qquad (*)$$

For example, since $-63 \in B(5,7)$, then $B(5,7) = B(5,2)$.

For the given $r \in B(n,a) \cap B(q,d)$, we now consider the set $B(nq,m)$ where m is the least natural number such that

$$r = tn + a = hq + d = s(qn) + m \ \in \ B(nq,m)$$

for $t, h \in \mathbb{Z}$ and a specific value of $s \in \mathbb{Z}$. That is, $m \in \{0,1,2,\ldots,qn-1\}$.

So we have $r \in B(nq,m)$. If we can show that $B(nq,m) \subseteq B(n,a) \cap B(q,d)$, then we have $r \in B(nq,m) \subseteq B(n,a) \cap B(q,d)$, the condition required to complete Step 2 and we are done.
Let $x \in B(nq,m)$. Then

$$\begin{array}{rcl} x = s(nq) + m & = & (sq)n + m \ \in \ B(n,m) \\ x = s(nq) + m & = & (sn)q + m \ \in \ B(q,m) \end{array}$$

So $x \in B(n,m) \cap B(q,m)$. So $r \in B(qn,m) \subseteq B(n,m) \cap B(q,m)$.
As shown above at $(*)$, since $r \in B(n,m)$ and $r \in B(q,m)$,

$$\begin{array}{rcl} B(n,m) & = & B(n,a_r) = B(n,a) \\ B(q,m) & = & B(q,d_r) = B(q,d) \end{array}$$

We then obtain the desired result,

$$r \in B(nq,m) \subseteq B(n,m) \cap B(q,m) \subseteq B(n,a) \cap B(q,d)$$

which allows us to conclude that the set $\mathscr{B}$ is a base for open sets of some topology on $\mathbb{Z}$.