Axioms and Set Theory

Link to textbook Axioms and Set Theory

This book is designed for students at the intermediate undergraduate level studying an introductory course in set theory. Although some experience in three or four first and second year university level course such as calculus and linear algebra is helpful, it is not essential since we will rarely refer to concepts learned in such courses. Some experience applying various forms of mathematical reasoning in any field is more important. Experience at handling functions in a variety of ways, as taught in high-school, is also useful. It is written with curious and inquiring students in mind; the text speaks more specifically to readers who have a strong desire to understand what doing mathematics is about. Those students who are comfortable with asking questions such as ``Why is that true?'' or responding ``I hear what you are saying but I still don't see it'' will benefit most from the pedagogical style adopted for this textbook.

It is also important to understand what this book is not. It not a self-study book. It assumes continuous interaction with an instructor who ensures that the students are not led astray by various misinterpretations of a given concept. Those who have studied a lot of mathematics know that, we often learn by first getting it wrong, then getting it wrong again, then understanding why it is wrong, and finally getting it right. Which makes the end result all the more satisfying. It is not a ``Set theory for dummies'' book nor a ``Master set theory in ten days'' textbook. There is a substantial amount of material in this book. No claims that set theory is easy are made. However, there is a hope that students will eventually develop the mathematical skills and knowledge to better understand how modern mathematics is done. If, in the end, the student is more mathematically literate then when he or she has started at least one of the many important objectives has been attained.

The course begins with an informal discussion of primitive concepts and the ZFC- axioms. We then discuss, in this order, operations on classes and sets, relations on classes and sets, functions, construction of numbers (beginning with the natural numbers followed by the rationals and reals), infinite sets, cardinal numbers and finally ordinal numbers. It is hoped that the reader will eventually understand why the ordinal numbers are viewed by many authors as the ``spine of set theory''. Towards the end of the book the Well-ordering theorem and Zorn's lemma are proved to be equivalent forms of the Axiom of choice, one of the most controversial axioms of mathematics. It is only towards the end of the text that we briefly discuss the Axiom of regularity.

The level of abstraction increases in the sections of the book where the students are introduced to the concept of ordinal numbers. It is hoped that everything which is presented before this point will allow the students to grasp the essence of this fundamental part of set theory.

As the student progresses through the course she or he is assumed to develop a better understanding of what constitutes a correct mathematical proof. To help attain this objective numerous examples of simple straightforward proofs are presented as models throughout the text.

The subject material is subdivided into nine major parts. Initially, these parts are themselves subdivided in ``bite-size'' sections, making it easier for students to test their understanding on fewer notions at a time. This will allow the instructor to better diagnose those specific points which challenge the students the most, thus helping to eliminate obstacles which may slow down their progression down the line.

Towards the end of the textbook, chapter sections may require more time and thought to better understand underlying concepts.

Each chapter is followed by a list of *Concepts review*.

These are questions which help the students highlight the main ideas presented in that section and review the concepts before attempting the exercises.

The answers to all *Concept review* questions are in main body of the textbook. While answering these questions the student will often discover essential notions which were overlooked or neglected. Textbook examples will serve as solution models to most of the exercise question at the end of each section. Exercise questions are divided in three groups A, B and C. The answers to the group A questions normally follow immediately from definitions and theorem statements presented in the text. The group B questions require a deeper understanding of the concepts, while the group C questions allow the students to deduce by themselves a few consequences of theorem statements presented in the text.

Readers who already possess a substantial amount of mathematical background may feel they can comfortably skip many chapters of the textbook without loss of continuity, since many chapters contain material they are already familiar with. The following sequence will allow readers with the required background to advance more quickly to the meat of the textbook: Chapters 1 on the topic of the ZFC-axioms, chapters 13 and 14 on the topic of natural numbers, chapters 18 to 22 on the topic of infinite sets and cardinal numbers, chapters 26 to 29 on ordinals and cardinals, and finally, chapters 30 and 31 on the axiom of choice, the axiom of regularity and Martin's axiom.

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A link to the pdf format of the book:

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